(ψ − α)-Meir-Keeler-Khan Type Fixed Point Theorem in Partial Metric Spaces

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ABSTRACT: In this paper, we introduce a new concept of (ψ − α)-Meir-Keeler-Khan type mappings in partial metric spaces. The presented theorems generalize and improve many existing results in the literature. Moreover, an examples is given to illustrate our results.

Key Words: (ψ − α)-Meir-Keeler-Khan mappings, partial metric spaces, fixed point.

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1. Introduction

In 1978, Fisher [1] proved the following revised version of result of Khan[2].

Theorem 1.1: ([1]) Let (X, d) be a metric space and f be a self map on X satisfying the following:

\[ d(fx, fy) \leq k \frac{d(x, fx)d(x, fy) + d(x, fy)d(y, fx)}{d(x, fy) + d(y, fx)}, \quad k \in [0, 1), \]

if \( d(x, fy) + d(y, fx) \neq 0, \)

and

\[ d(fx, fy) = 0 \quad \text{if} \quad d(x, fy) + d(y, fx) = 0. \]

Then f has a unique fixed point \( t \in X. \) Moreover, for every \( t_0 \in X, \) the sequence \( \{f^nt_0\} \) converges to t.

In the sequel, \( \Psi \) denotes the family of all (c)-comparison functions. A self map \( \psi \) on \( [0, \infty) \) is said to be a (c)-comparison function, if \( \sum_{n=1}^{\infty} \psi^n(t) < \infty \) for each \( t > 0, \) where \( \psi^n \) is the nth iterate of \( \psi. \) Clearly, \( \psi(0) = 0 \) and \( \psi(t) < t \) for all \( t > 0. \)

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Recently, Samet et al. [3] introduced the notion of $\alpha$-admissible mappings as follows:

**Definition 1.2:** ([3]) Let $f$ be a self map on $X$ and $\alpha : X^2 \to [0, \infty)$. If $\alpha(x, y) \geq 1 \Rightarrow \alpha(fx, fy) \geq 1$, for all $x, y \in X$, then $f$ is said to be $\alpha$-admissible.

One can refer [4-5] for class of $\alpha$-admissible mappings and more information on subject.

Matthews [6] introduced the notion of partial metric spaces as follows:

Let $X$ be a nonempty set and $p : X^2 \to [0, \infty)$ satisfy the following:

$(pm1)$ $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y);
$(pm2)$ $p(x, x) \leq p(x, y);
$(pm3)$ $p(x, y) = p(y, x);
$(pm4)$ $p(x, y) \leq p(x, z) + p(z, y) - p(z, z),

for all $x, y, z \in X$. Then $p$ is called a partial metric and the pair $(X, p)$ is called a partial metric space.

We note that the function $d_p : X \times X \to \mathbb{R}^+$ defined by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

satisfies the conditions of a metric space $X$ and hence it is a usual metric on $X$.

**Definition 1.3:** [6]

(i) A sequence $\{x_n\}$ in the PMS $(X, p)$ converges to $x$ if and only if $p(x, x) = \lim_{n \to \infty} p(x, x_n)$.

(ii) A sequence $\{x_n\}$ in the PMS $(X, p)$ is called a Cauchy sequence if $\lim_{n, m \to \infty} p(x_n, x_m)$ exists and is finite.

(iii) A PMS $(X, p)$ is called complete, if every Cauchy sequence $\{x_n\}$ in $X$ converges.

The following Lemma will be used in the sequel.

**Lemma 1.4:** [6]

1. A sequence $\{x_n\}$ is a Cauchy sequence in the PMS $(X, p)$ if and only if it is a Cauchy sequence in the metric space $(X, d_p)$.

2. A partial metric space $(X, p)$ is complete if and only if the metric space $(X, d_p)$ is complete. Moreover

$$\lim_{n \to \infty} d_p(x, x_n) = 0 \Leftrightarrow p(x, x) = \lim_{n \to \infty} p(x, x_n) = \lim_{n, m \to \infty} p(x_n, x_m).$$

In 1969, Meir and Keeler [7] proved an interesting fixed point theorem on a metric space $(X, d)$. Further, Redjel et al. [8] introduced the concept of $(\alpha - \psi)$-Meir-
Keeler-Khan mappings in metric spaces.

2. Main Results

In this section, we introduce a new concept of \((\psi - \alpha)\)-Meir-Keeler-Khan mappings in partial metric spaces and we establish a fixed point theorem via \(\alpha\)-admissible mappings. In the sequel, we consider that if \(T : X \to X\), then for all \(x, y \in X, x \neq y \Rightarrow p(x, Ty) + p(y, Tx) \neq 0\).

Definition 2.1: Let \((X, p)\) be a partial metric space, \(T : X \to X\) and \(\psi \in \Psi\). Then \(T\) is called a generalized Meir-Keeler-Khan type \(\psi\)-contraction whenever for each \(\epsilon > 0\), there exists \(\delta > 0\) such that

\[
\epsilon \leq \psi(p(x, Tx)p(x, Ty) + p(y, Ty)p(y, Tx)) < \epsilon + \delta(\epsilon) \Rightarrow p(Tx, Ty) < \epsilon.
\]

Definition 2.2: Let \((X, p)\) be a partial metric space, \(T : X \to X\), \(\psi \in \Psi\) and \(\alpha : X^2 \to [0, \infty)\). Then \(T\) is called a generalized Meir-Keeler-Khan type \(\psi - \alpha\)-contraction if the following conditions are satisfied:
(i) \(T\) is \(\alpha\) admissible;
(ii) for each \(\epsilon > 0\), there exists \(\delta > 0\) such that

\[
\epsilon \leq \psi\left(\frac{p(x, Tx)p(x, Ty) + p(y, Ty)p(y, Tx)}{p(x, Ty) + p(y, Tx)}\right) < \epsilon + \delta(\epsilon) \Rightarrow \alpha(x, x) \alpha(y, y)p(Tx, Ty) < \epsilon.
\]

Remark 2.3: It is clear that if \(T : X \to X\) be an \(\psi - \alpha\)-Meir-Keeler-Khan type mapping then

\[
\alpha(x, x) \alpha(y, y)p(Tx, Ty) \leq \psi\left(\frac{p(x, Tx)p(x, Ty) + p(y, Ty)p(y, Tx)}{p(x, Ty) + p(y, Tx)}\right),
\]

for all \(x, y \in X\).

Theorem 2.4: Let \((X, p)\) be a complete partial metric space and \(\psi \in \Psi\). If \(\alpha : X^2 \to \mathbb{R}^+\) satisfies the following conditions:
(i) there exists \(x_0 \in X\) such that \(\alpha(x_0, x_0) \geq 1\);
(ii) if \(\alpha(x_k, x_k) \geq 1\) for all \(k \in \mathbb{N}\), then \(\lim_{k \to \infty} \alpha(x_k, x_k) \geq 1\);
(iii) \(\alpha : X^2 \to \mathbb{R}^+\) is a continuous function in each coordinate.
Suppose that \(T : X \to X\) is a generalized Meir-Keeler-Khan type \(\psi - \alpha\)-contraction. Then \(T\) has a fixed point in \(X\).

Proof: Let \(x_0 \in X\) and \(x_{k+1} = Tx_k = T^kx_0\), for \(k = 0, 1, 2, 3, \ldots\). Since \(T\) is \(\alpha\)-admissible and \(\alpha(x_0, x_0) \geq 1\), we have
\[ \alpha(Tx_0, Tx_0) = \alpha(x_1, x_1) \geq 1. \]

Proceeding in the same manner, we get

\[ \alpha(x_k, x_k) \geq 1, \quad (2.3) \]

for all \( k \in \mathbb{N} \cup \{0\} \).

If \( x_{k_0+1} = x_{k_0} \) for some \( k_0 \in \mathbb{N} \), then \( x_{k_0} \) is the fixed point of \( T \). So, we suppose that \( x_{k+1} \neq x_k \) for all \( k \in \mathbb{N} \cup \{0\} \).

Using the definition of \( \psi \), we have

\[ \psi\left( \frac{p(x_k, Tx_k)p(x_k, Tx_{k+1}) + p(x_{k+1}, Tx_{k+1})p(x_{k+1}, Tx_k)}{p(x_k, Tx_{k+1}) + p(x_{k+1}, Tx_k)} \right) > 0, \]

for all \( k \in \mathbb{N} \cup \{0\} \).

We shall assert that

\[ \lim_{k \to \infty} p(x_k, x_{k+1}) = 0, \quad i.e., \quad \lim_{k \to \infty} d_p(x_k, x_{k+1}) = 0. \]

From (2) and (3), we have

\[ p(x_{k+1}, x_{k+2}) = p(Tx_k, Tx_{k+1}) \]
\[ \leq \alpha(x_k, x_k)\alpha(x_{k+1}, x_{k+1})p(Tx_k, Tx_{k+1}) \]
\[ < \psi\left( \frac{p(x_k, Tx_k)p(x_k, Tx_{k+1}) + p(x_{k+1}, Tx_{k+1})p(x_{k+1}, Tx_k)}{p(x_k, Tx_{k+1}) + p(x_{k+1}, Tx_k)} \right) \]
\[ = \psi\left( \frac{p(x_k, x_{k+1})p(x_k, x_{k+2}) + p(x_{k+1}, x_{k+2})p(x_{k+1}, x_{k+1})}{p(x_k, x_{k+2}) + p(x_{k+1}, x_{k+1})} \right) \]
\[ < p(x_k, x_{k+1}), \quad (2.4) \]

If \( p(x_k, x_{k+1}) \leq p(x_{k+1}, x_{k+2}) \), then

\[ p(x_{k+1}, x_{k+2}) = \psi\left( \frac{p(x_{k+1}, x_{k+2})p(x_k, x_{k+2}) + p(x_{k+1}, x_{k+2})p(x_{k+1}, x_{k+1})}{p(x_k, x_{k+2}) + p(x_{k+1}, x_{k+1})} \right) \]
\[ = \psi(p(x_{k+1}, x_{k+2})) \]
\[ < p(x_{k+1}, x_{k+2}), \]

which is a contradiction, and hence \( p(x_k, x_{k+1}) < p(x_{k-1}, x_k) \).

Using the same argument as above, we have for each \( n \in \mathbb{N} \),

\[ p(x_{k+1}, x_{k+2}) = p(Tx_k, Tx_{k+1}) \]
\[ \leq p(x_k, x_{k+1}). \quad (2.5) \]

Since the sequence \( \{p(x_k, x_{k+1})\} \) is decreasing, it must converge to some \( \epsilon \geq 0 \),

that is,
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\[
\lim_{k \to \infty} p(x_k, x_{k+1}) = \epsilon. \tag{2.6}
\]

From (5) and (6), we have

\[
\lim_{k \to \infty} \psi(p(x_k, x_{k+1})) = \epsilon. \tag{2.7}
\]

Here \( \epsilon = \inf \{p(x_k, x_{k+1}) : k \in \mathbb{N}\} \). We assert that \( \epsilon = 0 \). On the contrary, suppose that, \( \epsilon > 0 \). Since \( T \) is a generalized Meir-Keeler-Khan type \( \psi - \alpha \)-contraction, corresponding to \( \epsilon \) use, and using (7), there exists \( \delta > 0 \) and a natural number \( n \) such that

\[
\epsilon \leq \psi(p(x_n, x_{n+1})) < \epsilon + \delta \Rightarrow \alpha(x_n, x_n)\alpha(x_{n+1}, x_{n+1})p(Tx_n, Tx_{n+1}) < \epsilon,
\]

implies that,

\[
p(x_{n+1}, x_{n+2}) = p(Tx_n, Tx_{n+1}) \leq \alpha(x_n, x_n)\alpha(x_{n+1}, x_{n+1})p(Tx_n, Tx_{n+1}) < \epsilon,
\]

which is a contradiction, since

\[
\epsilon = \inf \{p(x_k, x_{k+1}) : k \in \mathbb{N}\}.
\]

Thus, we have that

\[
\lim_{k \to \infty} p(x_k, x_{k+1}) = 0. \tag{2.8}
\]

Also, from (pm2), we have

\[
\lim_{k \to \infty} p(x_k, x_k) = 0. \tag{2.9}
\]

Since \( d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y) \), for all \( x, y \in X \), using (8) and (9), we get

\[
\lim_{k \to \infty} d_p(x_k, x_{k+1}) = 0. \tag{2.10}
\]

Now, we assert that \( \{x_k\} \) is a Cauchy sequence in the partial metric space \((X, p)\). To show, it is sufficient to that \( \{x_k\} \) is a Cauchy sequence in the metric space...
On the contrary, let us suppose \( \{x_k\} \) is not a Cauchy sequence. So, there exists \( \eta > 0 \) such that for any \( c \in \mathbb{N} \), there are \( n_c, m_c \in \mathbb{N} \) with \( n_c > m_c \geq c \) satisfying

\[
d_p(x_{m_c}, x_{n_c}) \geq \eta. \tag{2.11}
\]

Also, for \( m_c \geq c \), we can choose a smallest positive integer \( n_c \) such that \( n_c > m_c \geq c \) and \( d(x_{2m_c}, x_{2n_c}) \geq \eta \). Therefore, we have

\[
d_p(x_{m_c}, x_{n_c} - 2) < \eta. \tag{2.12}
\]

Now, we have that for all \( c \in \mathbb{N} \),

\[
\eta \leq d_p(x_{m_c}, x_{n_c}) \\
\leq d_p(x_{m_c}, x_{n_c} - 2) + d_p(x_{n_c} - 2, x_{n_c} - 1) + d_p(x_{n_c} - 1, x_{n_c}) \\
< \eta + d_p(x_{n_c} - 2, x_{n_c} - 1) + d_p(x_{n_c} - 1, x_{n_c}). \tag{2.13}
\]

Letting \( c \to \infty \), we get

\[
\lim_{c \to \infty} d_p(x_{m_c}, x_{n_c}) = \eta. \tag{2.14}
\]

On the other hand, we have

\[
\eta \leq d_p(x_{m_c}, x_{n_c}) \\
\leq d_p(x_{m_c}, x_{m_c + 1}) + d_p(x_{m_c + 1}, x_{n_c + 1}) + d_p(x_{n_c + 1}, x_{n_c}) \\
\leq d_p(x_{m_c}, x_{m_c + 1}) + d_p(x_{m_c + 1}, x_{m_c}) + d_p(x_{m_c}, x_{n_c}) \\
+ d_p(x_{n_c}, x_{n_c + 1}) + d_p(x_{n_c + 1}, x_{n_c}).
\]

Letting \( c \to \infty \), we get

\[
\lim_{c \to \infty} d_p(x_{m_c + 1}, x_{n_c + 1}) = \eta. \tag{2.15}
\]

Since \( d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y) \) and using (14) and (15), we have that

\[
\lim_{c \to \infty} d_p(x_{m_c}, x_{n_c}) = \eta. \tag{2.16}
\]
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and

\[ \lim_{c \to \infty} d_p(x_{m_c+1}, x_{n_c+1}) = \frac{\eta}{2}. \tag{2.17} \]

From (2), we have

\[
p(x_{m_c+1}, x_{n_c+1}) = p(Tx_{m_c}, Tx_{n_c}) \\
\leq \alpha(x_{m_c}, x_{m_c}) \alpha(x_{n_c}, x_{n_c}) p(Tx_{m_c}, Tx_{n_c}) \\
< \psi(\frac{p(x_{m_c}, Tx_{m_c}) + p(x_{n_c}, Tx_{n_c})}{p(x_{m_c}, Tx_{n_c}) + p(x_{n_c}, Tx_{m_c})}) \\
< \psi(\frac{p(x_{m_c}, x_{m_c+1}) + p(x_{n_c}, x_{n_c+1}) + p(x_{n_c}, x_{m_{c+1}})}{p(x_{m_c}, x_{n_c+1}) + p(x_{n_c}, x_{m_{c+1}})}) . \tag{2.18} \]

Since,

\[
p(x_{m_c}, x_{n_c+1}) \leq p(x_{m_c}, x_{m_{c+1}}) + p(x_{m_{c+1}}, x_{n_c+1}) - p(x_{m_c+1}, x_{m_{c+1}}), \tag{2.19} \]

and

\[
p(x_{n_c}, x_{m_{c+1}}) \leq p(x_{n_c}, x_{n_{c+1}}) + p(x_{n_{c+1}}, x_{m_{c+1}}) - p(x_{n_{c+1}}, x_{n_{c+1}}). \tag{2.20} \]

Using (9), (18), (19) and (20) and making \( c \to \infty \), we have

\[ \frac{\eta}{2} < \psi(\frac{\eta}{2}) \leq \frac{\eta}{2}, \]

a contradiction.

Hence \( \{x_k\} \) is a Cauchy sequence in the metric space \((X, d_p)\).

Now, we assert that \( T \) has a fixed point \( z \).

Since \((X, p)\) is complete, so by Lemma 1.4, \((X, d_p)\) is also complete. Thus, there exists \( z \in X \) such that \( \lim_{k \to \infty} d_p(x_k, z) = 0 \). Moreover, from Lemma 1.4, we have

\[
p(z, z) = \lim_{k \to \infty} p(x_k, z) = \lim_{k, l \to \infty} p(x_k, x_l). \tag{2.21} \]

Further, since \( \{x_k\} \) is a Cauchy sequence in the metric space \((X, d_p)\), so \( \lim_{k \to \infty} d_p(x_k, x_l) = 0 \).

Since, \( d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y) \), we get

\[
\lim_{k, l \to \infty} p(x_k, x_l) = 0. \tag{2.22} \]
From (21) and (22), we have
\[ p(z, z) = \lim_{k \to \infty} p(x_k, z) = \lim_{k \to \infty} p(x_k, z) = 0. \]

Again, from (2), we get
\[
\begin{align*}
p(x_{k+1}, Tz) &= p(Tx_k, Tz) \\
&\leq \alpha(x_k, x_k)\alpha(z, z)p(Tx_k, Tz) \\
&\leq \psi\left(\frac{p(x_k, Tx_k)p(x_k, Tz) + p(z, Tz)p(z, Tx_k)}{p(x_k, Tz) + p(z, Tx_k)}\right) \\
&= \psi\left(\frac{p(x_k, x_{k+1})p(x_k, Tz) + p(z, Tz)p(z, x_{k+1})}{p(x_k, Tz) + p(z, x_{k+1})}\right).
\end{align*}
\]

Making \( k \to \infty \), we get
\[ p(z, Tz) \leq \psi(0) = 0, \]
that is, \( Tz = z \).

**Corollary 2.5.** Let \((X, p)\) be a partial metric space and \( \psi \in \Psi \). Suppose that \( T : X \to X \) is a generalized Meir-Keeler-Khan type \( \psi \)-contraction. Then \( T \) has a fixed point in \( X \).

**Proof.** By putting \( \alpha(x, y) = 1 \) in Theorem 2.4, we get the result.

**Example 2.6.** Let \( X = [0, 1] \) and \( p(x, y) = \max\{x, y\} \), then \((X, p)\) is a partial metric space. Define \( \alpha : [0, 1]^2 \to \mathbb{R}^+ \) by \( \alpha(x, y) = 1 + x + y \), and \( T : X \to X \) by \( Tx = \frac{x}{2} \). Also, let \( \psi : [0, \infty) \to [0, \infty) \) be defined by \( \psi(t) = \frac{t}{2} \). Clearly, \( T \) is \( \alpha \)-admissible. Without loss of generality, assume that \( x \geq y \). Then for all \( x, y \in [0, 1] \), we have \( \alpha(x, x)\alpha(y, y)p(Tx, Ty) \geq \frac{x}{2} \). Now, \( p(x, x) = p(y, Ty) = y, p(x, Ty) = y, p(x, Ty) = p(x, \frac{y}{2}) = x, p(y, Ty) = p(y, \frac{x}{2}) \).

**Case 1.** If \( p(y, \frac{x}{2}) = y \), then
\[
\psi\left(\frac{p(x, Tx)p(x, Ty)p(y, Ty)}{p(x, Ty) + p(y, Ty)}\right) = \psi\left(\frac{x^2 + y^2}{x + y}\right) = \psi\left(\frac{x^2 + y^2}{2(x + y)}\right) \leq \frac{x^2}{4} \leq \frac{x^2}{2}.
\]

**Case 2.** If \( p(y, \frac{x}{2}) = \frac{x}{2} \), then
\[
\psi\left(\frac{p(x, Tx)p(x, Ty)p(y, Ty)}{p(x, Ty) + p(y, Ty)}\right) = \psi\left(\frac{x^2 + y^2}{x + y}\right) = \psi\left(\frac{x^2 + y^2}{2(x + y)}\right) \leq \frac{x^2}{4} = \frac{x^2}{4}.
\]

Hence all the conditions of Theorem 2.4 are satisfied and 0 is the fixed point of \( T \).

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