Valuation Derived from Graded Ring and Module and Krull Dimension Properties

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ABSTRACT: In this paper we show if $R$ is a graded ring then we can define a valuation on $R$ induced by graded structure, and we prove some properties and relations for $R$. Later we show that if $R$ is a graded ring and $M$ a graded $R$-module then there exists a valuation on of $M$ which is derived from graded structure and also we prove some properties and relations for $R$. In the following we give a new method for finding the Kurll dimension of a valuation ring.

Key Words: Graded Ring, Graded Module, Valuation Ring, Valuation on a Module, Krull Dimension.

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1. Introduction

In the algebra valuation ring and graded ring are two most important structures [10], [11], [12]. We know that the graded ring is also the most important structure since the graded ring is a base for dimension and [4], on the depth of the associated graded ring of a filtration [5], [6]. So, as these important structures, the relation between these structures are useful for finding some new structures, and if $R$ is a discrete valuation ring then $R$ has many properties that have many uses for example Decidability of the theory of modules over commutative valuation domains [12], Rees valuations and asymptotic primes of rational powers in Noetherian rings and lattices [11]. Graded ring is also most important structures especially for making some non-commutative structures with commutative rings [3].

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In [1] M.H. Anjom SHoa and M.H. Hosseini show that there exists a filtered 
ring which derived from it and they find some new properties. Consequently M.H. 
Anjom SHoa and M.H. Hosseini in [2] proved that there exists a quasi valuation 
ring derived from filtered ring and also they proved if \( R \) is a strongly filtered ring 
then we can find a valuation ring which derived from \( R \). In this article they find 
some new properties.

In this article we investigate the relation between graded ring and graded Module. 
We prove that if we have graded ring then we can find a valuation or quasi valuation 
ring. Continuously We prove that if \( M \) be a graded module then we can define a 
valuation on \( M \). In following we prove a method for finding Krull dimension.

2. Preliminaries

**Definition 2.1.** A filtered ring \( R \) is a ring together with a family \( \{ R_n \}_{n \geq 0} \) of 
subgroups of \( R \) satisfying in the following conditions:

i) \( R_0 = R \);

ii) \( R_{n+1} \subseteq R_n \) for all \( n > 0 \);

iii) \( R_n R_m \subseteq R_{n+m} \) for all \( n, m > 0 \).

If \( R \) has conditions (i) , (ii) and instead of (iii) it has the following condition: 
\( R_n R_m = R_{n+m} \) for all \( n, m > 0 \) then we say \( R \) has a strong filtration.

**Definition 2.2.** Let \( R \) be a ring and \( I \) an ideal of \( R \). Then \( R_n = I^n \) is called 
\( I \)-adic filtration.

**Definition 2.3.** Let \( R \) be a filtered ring with filtration \( \{ R_n \}_{n \geq 0} \) and 
\( M \) be a \( R \)-module with family \( \{ M_n \}_{n \geq 0} \) of subgroups of \( M \) satisfying the following condi-
tions:

i) \( M_0 = M \);

ii) \( M_{n+1} \subseteq M_n \) for all \( n \geq 0 \);

iii) \( R_n M_m \subseteq M_{n+m} \) for all \( n, m \geq 0 \).

Then \( M \) is called filtered \( R \)-module.

**Definition 2.4.** Let \( R \) be a filtered ring with filtration \( \{ R_n \}_{n \geq 0} \) and \( M \) be a 
\( R \)-module with family \( \{ M_n \}_{n \geq 0} \) of subgroups of \( M \) satisfying the following condi-
tions:

i) \( M_0 = M \);

ii) \( M_{n+1} \subseteq M_n \) for all \( n \geq 0 \);

iii) \( R_n M_m \subseteq M_{n+m} \) for all \( n, m \geq 0 \).

Then \( M \) is called filtered \( R \)-module.

**Definition 2.5.** Let \( R \) be a filtered ring with filtration \( \{ R_n \}_{n \geq 0} \) and \( M \) be a 
\( R \)-module together with a family \( \{ M_n \}_{n \geq 0} \) of subgroups of \( M \) satisfying the fol-
lowing conditions:
i) \( M_0 = R; \)

ii) \( M_{n+1} \subseteq M_n \) for all \( n \geq 0; \)

iii) \( R_n M_m = M_{n+m} \) for all \( n, m \geq 0, \)

Then we say \( M \) has a strong filtration.

Definition 2.7. Let \( R \) be a graded ring. An \( R \)-module is called a graded \( R \)-module if \( M \) can be expressed as a direct sum of subgroups \( \{ M_i \} \) i.e. \( M = \sum_{n \geq 0} M_n \) such that \( R_n M_m = M_{n+m} \) for all \( n, m \geq 0. \)

Definition 2.8. The elements of \( R_n \) or \( M_n \) in a graded ring or a module are called elements of the homogeneous degree \( n. \)

Definition 2.9. Let \( \Delta \) be a totally ordered abelian group. A quasi valuation \( \nu \) on \( R \) with values in \( \Delta \) is a mapping \( \nu: R^* \rightarrow \Delta \) satisfying :

i) \( \nu(ab) \geq \nu(a) + \nu(b); \)

ii) \( \nu(a + b) \geq \min \{ \nu(a), \nu(b) \}. \)

Definition 2.10. Let \( \Delta \) be a totally ordered abelian group. A valuation \( \nu \) on \( R \) with values in \( \Delta \) is a mapping \( \nu: R^* \rightarrow \Delta \) satisfying :

i) \( \nu(ab) = \nu(a) + \nu(b); \)

ii) \( \nu(a + b) \geq \min \{ \nu(a), \nu(b) \}. \)

Definition 2.11. Let \( M \) be an \( R \)-module where \( R \) is a ring, and \( \Delta \) an order set with maximum element \( \infty \) and \( \Delta \neq \{ \infty \}. \) A mapping \( \nu \) of \( M \) onto \( \Delta \) is called a valuation on \( M, \) if the following conditions are satisfied:

i) For any \( x, y \in M, \nu(x + y) \geq \min \{ \nu(x), \nu(y) \}; \)

ii) If \( \nu(x) \leq \nu(y), \) \( x, y \in M, \) then \( \nu(ax) \leq \nu(ay) \) for all \( a \in R; \)

iii) Put \( \nu^{-1}(\infty) := \{ x \in M | \nu(x) = \infty \}. \) If \( \nu(az) \leq \nu(bz), \) where \( a, b \in R, \) and \( z \in M \setminus \nu^{-1}(\infty), \) then \( \nu(ax) \leq \nu(bx) \) for all \( x \in M \)

iv) For every \( a \in R \setminus (\nu^{-1}(\infty) : M), \) there is an \( a' \in R \) such that \( \nu((a')x) = \nu(x) \) for all \( x \in M \)

3. Valuation on Graded Ring and Module

Let \( R \) be a ring with unit and \( R \) a graded ring with subgroups \( \{ R_n \}_{n \geq 0}. \)

Remark 3.1. Let \( R \) be a graded ring then for any nonzero \( r \in R \) has a unique expression as a sum of homogeneous elements, \( r = \sum_{n \in \mathbb{Z}} r_n \) where \( r_n \) is nonzero for a finite number of \( n \) in \( \mathbb{Z}. \)

Proposition 3.2. Let \( R \) be a graded ring \( R_0 \) is a subring of \( R \) and \( 1 \in R_0. \)
Proof: See ([8]).

Remark 3.3. Let \( R \) be a graded ring it is obvious that for all \( i \in \mathbb{Z} \) we have \( 0 \in R_i \).

Theorem 3.4. Let \( R \) be a graded ring and \( \Delta \) be ordered semigroup. If we define \( \nu : R \rightarrow \Delta \cup \{\infty\} \) such that for every \( x \in R \) we have \( \nu(x) = \sum \deg r_i \) where \( r = \sum r_i \) and \( \deg r_i = i \). Then \( \nu \) is a quasi valuation on \( R \).

Proof:

1) By proposition(3.2) we have \( \nu(1) = 0 \)

2) By remake(3.3) we have \( \nu(0) = \infty \)

3) Let \( r_i, r_j \in R \) such that \( \nu(r_i) = i \) and \( \nu(r_j) = j \) then \( r_i + r_j \in R_i + R_j \) so \( \deg (r_i + r_j) \geq \deg (r_i) \) and \( \deg (r_i + r_j) \geq \deg (r_j) \). Now let \( \alpha = \Sigma r_i \) and \( \beta = \Sigma r_j \) then \( \deg (\alpha + \beta) \geq \deg (\alpha) \) and \( \deg (\alpha + \beta) \geq \deg (\beta) \). So \( \nu(\alpha + \beta) \geq \min \{\nu(\alpha), \nu(\beta)\} \)

4) Let \( r_i, r_j \in R \) such that \( \nu(r_i) = i \) and \( \nu(r_j) = j \) then \( r_i r_j \in R_i R_j \subseteq R_{i+j} \) there for \( \deg (r_i r_j) \geq i + j \).

Now let \( \alpha, \beta \in R \) and \( \alpha = \Sigma r_i \) and \( \beta = \Sigma r_j \) then \( \nu(\alpha \beta) = \nu(\alpha) + \nu(\beta) \)

By above condition \( \nu : R \rightarrow \Delta \cup \{\infty\} \) is a quasi valuation on \( R \).

Proposition 3.5. Let \( R \) be a strongly graded ring then \( \nu : R \rightarrow \Delta \cup \{\infty\} \) is a valuation on \( R \).

Proof: By theorem(3.4) \( R \) is a quasi valuation ring. Since \( R \) is a strongly graded ring then we have \( R_i R_j = R_{i+j} \) then \( \deg (r_i r_j) = i + j \) there for if \( \alpha, \beta \in R \) and \( \alpha = \Sigma r_i \) and \( \beta = \Sigma r_j \) then \( \nu(\alpha \beta) = \nu(\alpha) + \nu(\beta) \).

Theorem 3.6. Let \( R \) be a filtered ring with filtration \( \{R_n\}_{n \geq 0} \). Now we define \( \omega : R \rightarrow \mathbb{Z} \) such that for every \( \alpha \in R \) and \( \omega(\alpha) = \min \{i | \alpha \in R_i \setminus R_{i+1}\} \). Then \( \omega \) is a quasi valuation on \( R \).

Proof: See ([2])

Theorem 3.7. Let \( G(R) \) be an associated graded ring where \( R \) is a filtered ring. Then \( \nu : R \rightarrow \Delta \cup \{\infty\} \) is a quasi valuation on \( G(R) \) which derived from quasi valuation \( \omega \) on \( R \) as defined in theorem(3.6).

Proof: Since \( \omega : R \rightarrow \mathbb{Z} \) such that for every \( \alpha \in R \) and

\[
\omega(\alpha) = \min \{i | \alpha \in R_i \setminus R_{i+1}\}
\]

for every \( r_i \in G(R_i) \) we have \( \deg (r_i) = \omega(\alpha) \) where \( r_i = \alpha + R_{i+1} \).
Remark 3.8. If $R$ is a ring, we will denote by $Z(R)$ the set of zero-divisors of $R$ and by $T(R)$ the total ring of fractions of $R$.

Definition 3.9. A ring $R$ is said to be a Manis valuation ring (or simply a Manis ring) if there exists a valuation $\nu$ on its total fractions $T(R)$, such that $R = R_\nu$.

Definition 3.10. A ring $R$ is said to be a Prüfer ring if each overring of $R$ is integrally closed in $T(R)$.

Definition 3.11. A Manis ring $R_\nu$ is said to be $\nu$-closed if $R_\nu/\nu^{-1}(\infty)$ is a valuation domain (see Theorem 2 of [11]).

Corollary 3.12. Let $R$ be a strongly graded ring. Then $R$ is Manis ring.

Proposition 3.13. Let $R_\nu$ be a Manis ring. If $R_\nu$ is $\nu$-closed, then $R_\nu$ is Prüfer.

Proof: See proposition 1 of [12] □

Proposition 3.14. Let $R$ be a strongly graded ring. Then $R$ is $\nu$-closed.

Proof: By proposition (3.5) and definition (3.9) we have $R$ is a Manis ring and $R = R_\nu$.
Now let $\alpha, \beta \in R$ and $\nu(\alpha) = i$ and $\nu(\beta) = j$

Consequently if
$$(\alpha + \nu^{-1}(\infty))(\beta + \nu^{-1}(\infty)) \in \nu^{-1}(\infty)$$

Then $i + j \geq \infty$ so $\alpha \in \nu^{-1}(\infty)$ or $\beta \in \nu^{-1}(\infty)$. Hence by definition (3.11) $R$ is $\nu$-closed. □

Corollary 3.15. Let $R$ be a strongly graded ring. Then $R$ is Prüfer.

Proof: By proposition (3.14) $R$ is $\nu$-closed so by corollary (3.15) $R$ is Prüfer. □

Corollary 3.16. Let $R$ be a noetherian and strongly graded domain. Then $R$ is a Dedekind domain.

Proof: By corollary (3.15) $R$ is Prüfer domain so $R$ is Dedekind domain. □

Theorem 3.17. Let $R$ be a domain. Then the following conditions are equivalent.

i) $R$ is a Prüfer domain;

ii) Every tow-generated ideal of $R$ is invertible;

iii) $R_P$ is a valuation for every prime ideal $P$ of $R$;
iv) \( R_m \) is a valuation for every maximal ideal \( m \) of \( R \);

v) Each finitely-generated non-zero ideal \( I \) of \( R \) is a cancellation ideal, that is \( IJ = I K \) for ideals \( J, K \), implies \( J = K \);

vi) \( R \) is integrally closed and there exists integer \( n > 0 \) such that for every two elements \( a, b \in R \), \((a, b)^n = (a^n, b^n)\);

Proof: See [3], [4], [5].}

**Proposition 3.18.** Let \( R \) be a strongly graded domain. Then we have following condition for \( R \):

i) Every tow-generated ideal of \( R \) is invertible;

ii) \( R_P \) is a valuation for every prime ideal \( P \) of \( R \);

iii) \( R_m \) is a valuation for every maximal ideal \( m \) of \( R \);

iv) Each finitely-generated non-zero ideal \( I \) of \( R \) is a cancelation ideal, that is \( IJ = I K \) for ideals \( J, K \), implies \( J = K \);

v) \( R \) is integrally closed and there exists integer \( n > 0 \) such that for every two elements \( a, b \in R \), \((a, b)^n = (a^n, b^n)\);

Proof: By Corollary(3.15) \( R \) is a Prüfer domain so by Theorem(3.17) we have the conditions}

**Remark 3.19.** Let \( R \) be a graded ring and \( M \) a graded \( R \)-module, then for any nonzero \( a \in M \) has a unique expression as a sum of homogeneous elements, \( a = \sum_{n \in \mathbb{Z}} m_n \) where \( a_n \) is nonzero for a finite number of \( n \) in \( \mathbb{Z} \).

**Remark 3.20.** Let \( M \) be a graded ring it is obvious that for all \( i \in \mathbb{Z} \) we have \( 0 \in M_i \).

**Theorem 3.21.** Let \( R \) be a graded ring and \( M \) a graded \( R \)-module. Let \( \Delta \) be an order semigroup. Then \( \nu : M \to \Delta \cup \{\infty\} \) where \( \nu(a_i) = \deg(a_i) \) is a valuation on \( M \).

Proof:

i) Let \( a_i, a_j \in M \) such that \( \nu(a_i) = i \) and \( \nu(a_j) = j \) then \( a_i + a_j \in M_i + M_j \)
so \( \deg(a_i + a_j) \geq \deg(a_i) \) and \( \deg(a_i + a_j) \geq \deg(a_j) \). Now let \( x = \Sigma a_i \) and \( y = \Sigma a_j \) then \( \deg(x + y) \geq \deg(x) \) and \( \deg(x + y) \geq \deg(y) \).
So \( \nu(x + y) \geq \min\{\nu(x), \nu(y)\} \)

ii) Let for \( x, y \in M \) we have \( \nu(x) = i \), \( \nu(y) = j \) and \( i \leq j \). For every \( r \in R \) where \( r \in R_k \) we have \( ax \in R_k M_i \subseteq M_{k+i} \) and \( ay \in R_k M_j \subseteq M_{k+j} \). There for \( \deg(ax) \geq k + i \) and \( \deg(ay) \geq k + j \) since \( i \leq j \) then \( k + i \leq k + j \) so for every \( a \in R \) we have \( \nu(ax) \leq \nu(ay) \)
iii) Let \( \nu(az) \leq \nu(bz) \), where \( a, b \in R \) such that \( a \in R_i \), \( b \in R_j \) and \( z \in M \setminus \nu^{-1}(\infty) \). Since \( z \in M \setminus \nu^{-1}(\infty) \) then there exists \( k \not= \infty \) such that \( z \in M_k \) and \( \nu(z) = k \). So we have \( i \leq j \). Now for every \( z \in M \) we have \( \nu(az) \leq \nu(bz) \).

iv) Let \( a \in R \setminus (\nu^{-1}(\infty) : M) \). If \( x \in \nu^{-1}(\infty) \) then for all \( a', a \in R \) \( \nu((a'a)x) = \nu(x) = \infty \).

Now let \( x \not\in \nu^{-1}(\infty) \) and for all \( a' \in R \) we have \( \nu((a'a)x) \neq \nu(x) \). Since \( 1 \in R_0 \) and for all \( x \in M \) we have \( 1x = x \) so we should have \( \deg(1) \neq 0 \) and it is contradiction.

\[ \blacksquare \]

**Proposition 3.22.** Let \( M \) be a graded \( R \)-module where \( R \) is a ring, and let \( \nu : M \to \Delta \) be a valuation on \( M \). Then the following statements are true:

i) If \( \nu(x) = \nu(y) \) for \( x, y \in M \), then \( \nu(ax) = \nu(ay) \) for all \( a \in R \);

ii) \( \nu(-x) = \nu(x) \) for all \( x \in M \);

iii) If \( \nu(x) \neq \nu(y) \), then \( \nu(x + y) = \min\{\nu(x), \nu(y)\} \);

iv) If \( \nu(az) = \nu(bz) \) for some \( a, b \in R \) and \( z \in M \setminus \nu^{-1}(\infty) \), then \( \nu(ax) = \nu(bx) \) for all \( x \in M \);

v) If \( \nu(az) < \nu(bz) \) for some \( a, b \in R \) and \( z \in M \), then \( \nu(ax) < \nu(bx) \) for all \( x \in M \setminus \nu^{-1}(\infty) \);

vi) The core \( \nu^{-1} \) of \( \nu \) is a prime submodule of \( M \);

vii) The following subsets constitute a valuation pair of \( R \) with core \( (M : \nu^{-1}(\infty)) \):

\[
A_{\nu} = \{ a \in A | \nu(ax) \geq \nu(x) \text{ for all } x \in M \},
\]

\[
P_{\nu} = \{ a \in A | \nu(ax) \geq \nu(x) \text{ for all } x \in M \setminus \nu^{-1}(\infty) \}
\]

**Proof:** see proposition 1.1 [6]

\[ \blacksquare \]

**Corollary 3.23.** If \( M \) is a graded \( R \)-module, then \( \nu : M \to \mathbb{Z} \) has all of the properties that explained in the Proposition (3.22).

**Proposition 3.24.** Let \( M \) be an \( R \)-module, where \( R \) is a ring. Then there is a valuation on \( M \), if and only if there exists a prime ideal \( P \) of \( R \) such that \( PM_P \neq M_P \), where \( M_P \) is the localization of \( M \) at \( P \).

**Proof:** see (Proposition 1.3 [6])

\[ \blacksquare \]
Corollary 3.25. Let $M$ be a graded $R$-module, where $R$ is a graded ring. Then there exists a prime ideal $P$ of $R$ such that $PM_P \neq M_P$, where $M_P$ is the localization of $M$ at $P$.

Proof: By theorem (3.21) there is a valuation on $M$, then by proposition (3.24) there exists a prime ideal $P$ of $R$ such that $PM_P \neq M_P$, where $M_P$ is the localization of $M$ at $P$. \qed

Definition 3.26. Let $M$ be an $R$-module where $R$ is a ring, and let $\nu$ be a valuation on $M$. A representation system of the equivalence relation $\sim_\nu$ is called a skeleton of $\nu$.

Definition 3.27. A subset $S$ of $M$ is said to be $\nu$-independent if $S \cap \nu^{-1}(\infty) = \emptyset$, and $\nu(x) \notin \nu(Ry)$ for any pair of distinct elements $x, y \in S$. Here, we adopt the convention that the empty subset $\emptyset$ is $\nu$-independent.

Corollary 3.28. Let $M$ be a graded $R$-module, where $R$ is a graded ring. Then there is a skeleton on $M$.

Proof: By theorem (3.21) there is a valuation on $M$, then by definition (3.26) we have there is a skeleton on $M$. \qed

Proposition 3.29. Let $M$ be a graded $R$-module where $R$ is a graded ring, and $\nu$ a valuation on $M$. If $\Lambda$ is a skeleton of $\nu$, then the following conditions are satisfied:

i) $\Lambda$ is a $\nu$-independent subset of $M$;

ii) For every $x \in M \setminus \nu^{-1}(\infty)$, there exists a unique $\lambda \in \Lambda$ such that $\nu(x) = \nu(R\lambda)$.

Proof: By corollary (3.28) $\Lambda$ is a skeleton of $\nu$ and by proposition (1.4, [6]) we have the above conditions. \qed

Proposition 3.30. Let $M$ be a graded $R$-module where $R$ is a graded ring, and $\nu$ a valuation on $M$. If $\Lambda$ is a skeleton of $\nu$. If $a_1\lambda_1 + \cdots + a_n\lambda_n = 0$ where $a_1, \cdots, a_n \in R$ and $\lambda_1 \cdots \lambda_n \in \Lambda$ are mutually distinct, then $a_i \in (\nu^{-1}(\infty) : M), i = 1, \cdots, n$.

Proof: By corollary (3.28) $\Lambda$ is a skeleton of $\nu$ and by proposition (1.5, [6]) we have If $a_1\lambda_1 + \cdots + a_n\lambda_n = 0$ where $a_1, \cdots, a_n \in R$ and $\lambda_1 \cdots \lambda_n \in \Lambda$ are mutually distinct, then $a_i \in (\nu^{-1}(\infty) : M), i = 1, \cdots, n$. \qed
4. Krull Dimension Derived from Valuation on Graded Ring and Module

**Definition 4.1.** Let \((E, \leq)\) be an order set. For \(a, b \in E\) we write \([a, b]\) for the set \(\{x \in E, a \leq x \leq b\}\) and we put \(\Gamma(E) = \{(a, b), a \leq b\}\).

**Remark 4.2.** By transfinite recurrence we define on \(\Gamma(E)\) the following filtration:

i) \(\Gamma^{-1}(E) = \{(a, b), a = b\}\);

ii) \(\Gamma_0 = \{(a, b) \in \Gamma(E), \text{[a,b] is Artinian}\}\);

iii) Supposing \(\Gamma_\beta\) has been defined for all \(\beta \leq \alpha\), then \(\Gamma_\alpha = \{(a, b) \in \Gamma(E), \forall b \geq b_1 \geq ... \geq b_n \geq ... \geq a, \text{there is an } n \in \mathbb{N} \text{ such that } [b_{i+1}, b_i] \in \Gamma_\beta(E) \text{ for all } i \geq n\}\).

**Remark 4.3.** We obtain an ascending chain \(\Gamma^{-1}(E) \subset \Gamma_0(E) \subset ... \subset \Gamma_\alpha(E) \subset ...\). There exists an ordinal \(\xi\) such that \(\Gamma_\xi(E) = \Gamma_{\xi+1}(E) = ...\).

**Definition 4.4.** If there exists an ordinal \(\alpha\) such that \(\Gamma(E) = \Gamma_\alpha(E)\) then \(E\) is said to have a Krull Dimension. The smallest ordinal with the property that \(\Gamma(E) = \Gamma_\alpha(E)\) will be called the Krull Dimension of \(E\) and we denoted by \(KdimE\).

**Definition 4.5.** A ring \(R\) is said to be a ring with an involution if there exists a mapping \(*: R \rightarrow R\) such that for every \(a, b \in R\):

i) \(a** = a\)

ii) \((a + b)* = a* + b*

iii) \((ab)^* = b*a^*

The symmetric element of \(R\) respect to (*) is the set \(\text{Sym}R = \{x \in R| x* = x\}\).

**Remark 4.6.** If \(R\) is associative with 1. \(\text{Sym}R\) will be the set of its symmetric elements. A subset \(P \subseteq R\) is called a \(*-\) Ordering if

i) \(P + P \subseteq P\)

ii) \(rPr^{-1} \subseteq P\) for all \(r \in R\)

iii) \(P \cup -P = \text{Sym}R\)

iv) \(P \cap -P = \{0\}\)

v) \(P\) is closed under the Jordan multiplication \(\{a, b\} = ab + ba\)

**Definition 4.7.** For every \(0 \leq \gamma \in \Delta\) we define a relation \(\sim_\gamma\) on \(R^*\) by \(x \sim_\gamma y :\Leftrightarrow \nu(x) + \gamma < \nu(x - y)\). We write \(\sim_0 : = \sim_0\).

**Lemma 4.8.** These relations are semigroup congruences and it is easy to see that they are cancellative and \(*\)-invariant.
Proof: See Lemma 1.5,1.6 in Da.

Definition 4.9. Let $R$ be a ring and $\Delta$ be order semigroup. If $\nu : R \rightarrow \Delta \cup \{\infty\}$ be a valuation on $R$ and $R^* = R \setminus \{0\}$.

i) $\nu$ is compatible with a $*$-ordering $P$ of $R$ iff $x \sim_{\nu} y \in P$ for all $x, y \in \text{Sym}R^*$.

ii) $\nu$ is called quasi-commutative iff for all $a, b \in R^*$ we have $ab \sim_{\nu} ba$.

iii) $\nu$ is weakly quasi-Ore iff for all $a, b \in R^*$ there exist $r, s \in R^*$ such that $ra \sim_{\nu} sb$. Note that this condition is left-right symmetric since $\nu$ is a $*-$valuation. Clearly, $\nu$ is weakly quasi-Ore iff the semigroup $R^*/\sim_{\nu}$ satisfies the Ore condition.

Proposition 4.10. Let $R$ be a ring and $\Delta$ be order semigroup. If we define $\nu : R \rightarrow \Delta \cup \{\infty\}$ such that $\nu$ is a quasi valuation on $R$, and for all $x, y \in R^*$ we define $x \sim_{\nu} y$ if and only if $\nu(x) = \nu(y)$ then:

i) $\nu$ is compatible with a $*$-ordering $P$ of $R$;

ii) $\nu$ is called quasi-commutative;

iii) $\nu$ is weakly quasi-Ore;

Proposition 4.11. Let $R$ be a ring and $\Delta$ be order semigroup. If we define $\nu : R \rightarrow \Delta \cup \{\infty\}$ such that $\nu$ is a quasi valuation on $R$ and $\nu(r)$ is defined as order $\leq$ for every $r \in R$. Then $\text{Sym}R$ is an order set.

Proof: Since $R$ is a quasi valuation, then by proposition 4.10 then $\text{Sym}R$ is an order set.

Corollary 4.12. Let $R$ be a ring and $\Delta$ be order semigroup. If we define $\nu : R \rightarrow \Delta \cup \{\infty\}$ such that $\nu$ is a quasi valuation on $R$ and $\nu(r)$ is defined as order $\leq$ for every $r \in R$. Then $\text{Sym}R$ has Krull Dimension.

Proof: By proposition 4.11 $R$ is order set so $R$ have Krull Dimension.

Corollary 4.13. Let $R$ be a ring and $\Delta$ be order semigroup. If we define $\nu : R \rightarrow \Delta \cup \{\infty\}$ such that $\nu$ is a quasi valuation on $R$ and $\nu(r)$ is defined as order $\leq$ for every $r \in R$. Then $\Gamma(-1)\Delta(SymR) = 0$.

Remark 4.14. Let $R$ be a ring and $\Delta$ be order semigroup. If we define $\nu : R \rightarrow \Delta \cup \{\infty\}$ such that $\nu$ is a quasi valuation on $R$ and $\nu(r)$ is defined as order $\leq$ for every $r \in R$. Then $K\dim R \leq K\dim \text{Sym}R$.

Example 4.15. Let $R$ be a DVR (Discreet valuation ring ), We know $K\dim R = 1$. Now we prove it by above structure. Since $R$ is DVR so $R$ is a quasi valuation ring.

i) $\Gamma(-1)(\text{Sym}R) = 0$;
ii) $\Gamma_0 = \{(a, b) \in \Gamma(R), [a, b] \text{is Artinian}\};$

iii) $\Gamma_1(\text{Sym}R)$ has been defined for all $\beta \leq 1$, then $\Gamma_1(R) = \{(a, b) \in \Gamma(\text{Sym}R), \forall b \geq b_1 \geq \ldots \geq b_n \geq \ldots \geq a, \text{there is an } n \in \mathbb{N} \text{ such that } [b_{i+1}, b_i] \in \Gamma_0(E) \text{ for all } i \geq n\}.$

There for as for every $0 \neq r_i \in R$ and $i \geq n$ we have $[r_{i+1}, r_i] \in \Gamma_{-1}$ or $[r_{i+1}, r_i] \in \Gamma_0$ and so $\Gamma_1(\text{Sym}R) = \Gamma(\text{Sym}R)$. Now by definition (4.4) we have $K\dim \text{Sym}R = 1$.

Now we have $K\dim R \leq 1$ since $K\dim R \neq 0$ so $K\dim R = 1$

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