On some metabelian 2-group and applications II

Abdelmalek Azizi, Abdelkader Zekhnini and Mohammed Taous

ABSTRACT: Let $G$ be some metabelian 2-group such that $G/G' \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. In this paper, we construct all the subgroups of $G$ of index 2 or 4, we give the abelianization types of these subgroups and we compute the kernel of the transfer map. Then we apply these results to study the capitulation problem of the 2-ideal classes of some fields $k$ satisfying the condition $\text{Gal}(k_{2}^{(2)}/k) \cong G$, where $k_{2}^{(2)}$ is the second Hilbert 2-class field of $k$.

Key Words: 2-group, metabelian 2-group, capitulation, Hilbert class fields.

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1. Introduction

Let $k$ be an algebraic number field and let $\text{Cl}(k)$ denote its class group. Let $k^{(1)}$ be the Hilbert class field of $k$, that is the maximal abelian unramified extension of $k$. Let $k^{(2)}$ be the Hilbert class field of $k^{(1)}$ and put $G = \text{Gal}(k^{(2)}/k)$. Denote by $F$ a finite extension of $k$ and by $H$ the subgroup of $G$ which fixes $F$, then we say that an ideal class of $k$ capitulates in $F$ if it is in $\ker j_{k \rightarrow F}$, the kernel of the homomorphism:

$$j_{k \rightarrow F} : \text{Cl}(k) \longrightarrow \text{Cl}(F)$$

induced by extension of ideals from $k$ to $F$. An important problem in Number Theory is to explicitly determine the kernel of $j_{k \rightarrow F}$, which is usually called the capitulation kernel. As $j_{k \rightarrow F}$ corresponds, by Artin reciprocity law, to the group theoretical transfer (for details see [14]):

$$V_{G \rightarrow H} : G/G' \longrightarrow H/H',$$

where $G'$ (resp. $H'$) is the derived group of $G$ (resp. $H$). So, determining $\ker j_{k \rightarrow F}$ is equivalent to determine $\ker V_{G \rightarrow H}$, which transforms the capitulation problem to a problem of Group Theory. That is why the capitulation problem is completely solved if $G/G' \cong (2, 2)$, since groups $G$ such that $G/G' \cong (2, 2)$ are determined and

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well classified (see \cite{11,14}). If $G/G' \simeq (2, 2^n)$, for some integer $n \geq 2$, then $G$ is metacyclic or not; in the first case the capitulation problem is completely solved, whereas in the second case the problem is open (see \cite{6,7}). If $G/G' \simeq (2, 2, 2)$, then the structure of $G$ is unknown in most cases, so the capitulation problem is also style open, in reality there are some studies which dealt with this problem in particular cases; see \cite{1,2,3,9,10}. It is the purpose of this paper to provide answers to this problem in a particular case, it is the continuation of a project we started in \cite{4,5}; we give some group theoretical results to solve the capitulation problem, in a particular case, if $G$ satisfies the last condition. For this, we consider the family of groups defined, for integers $n \geq 1$ and $m \geq 2$, as follows

$$G_{n,m} = \langle \sigma, \tau, \rho : \rho^4 = \tau^{2^{n+1}} = \sigma^{2^m} = 1, \rho^2 = \tau^{2^n} \sigma^{2^{m-1}}, [\tau, \sigma] = 1, [\rho, \sigma] = \sigma^2, [\rho, \tau] = \rho^2 \rangle$$

(1.1)

In this paper, we construct all the subgroups of $G_{n,m}$ of index 2 or 4, we give the abelianization types of these subgroups and we compute the kernel of the transfer map $V_{G\rightarrow H} : G_{n,m}/G_{n,m} \rightarrow H/H'$, for any subgroup $H$ of $G_{n,m}$, defined by the Artin map. Then we apply these results to study the capitulation problem of the 2-ideal classes of some fields $k$ satisfying the condition $\text{Gal}(k_2^{(2)}/k) \simeq G_{n,m}$, where $k_2^{(2)}$ is the second Hilbert 2-class field of $k$. Finally, we illustrate our results by some examples which show that our group is realizable i.e. there is a field $k$ such that $\text{Gal}(k_2^{(2)}/k) \simeq G_{n,m}$.

2. Main Results

Recall first that a group $G$ is said to be metabelian if its derived group $G'$ is abelian, and a subgroup $H$ of a group $G$, not reduced to an element, is called maximal if it is the unique subgroup of $G$ distinct from $G$ containing $H$.

Let $G_{n,m}$ be the group family defined by the Formula (1.1). Since $[\tau, \sigma] = 1$, $[\rho, \sigma] = \sigma^2$ and $[\rho, \tau] = \rho^2 = \tau^{2^n} \sigma^{2^{m-1}}$, so $G_{n,m} = \langle \sigma^2, \rho_2 \rangle = \langle \sigma^2, \tau^{2^n} \sigma^{2^{m-1}} \rangle = \langle \sigma^2, \tau^{2^n} \rangle$, which is abelian. Then $G_{n,m}$ is metabelian and $G_{n,m}/G_{n,m}' \simeq (2, 2, 2^n)$. Hence $G_{n,m}$ admits seven subgroups of index 2, denote them by $H_i,2$, and if $n = 1$ it admits also seven subgroups of index 4, we denote them by $H_i,4$, where $1 \leq i \leq 7$. These subgroups, their derived groups and the types of their abelianizations are given in Tables 1 and 2 below, where $b = \max(m, n + 1)$.

To check the Tables entries, we use the following lemmas.

**Lemma 2.1** ([12], Proposition 5.1.5). Let $x, y$ and $z$ be elements of some group $G$, put $x^y = y^{-1}xy$. Then $[xy, z] = [x, z]^y[y, z]$ and $[x, yz] = [x, z][x, y]^z$.

**Lemma 2.2**. Let $G_{n,m} = \langle \sigma, \tau, \rho \rangle$ denote the group defined above, then

1. $\rho^2$ commutes with $\sigma$ and $\tau$.
2. $\rho^{-1} \sigma \rho = \sigma^{-1}$.
3. $\tau^{-1} \rho \tau = \rho^3$ and $\rho^{-1} \tau \rho = \tau \rho^2$.  

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Table 1: Subgroups of $G_{n,m}$ of index 2

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$H_{i,2}$</th>
<th>$H'_{i,2}$</th>
<th>$H_{i,2}/H'_{i,2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\langle \sigma, \tau \rangle$</td>
<td>(1)</td>
<td>$(2^m, 2^{n+1})$</td>
</tr>
<tr>
<td>2</td>
<td>$\langle \sigma, \rho \rangle$</td>
<td>$\langle \sigma^2 \rangle$</td>
<td>(2, 4)</td>
</tr>
<tr>
<td>3</td>
<td>$m = 2$</td>
<td>$\langle \tau, \rho \rangle$</td>
<td>$\langle \rho^2 \rangle$</td>
</tr>
<tr>
<td></td>
<td>$m \geq 3$</td>
<td>$\langle \tau, \rho, \sigma \rangle$</td>
<td>$\langle \rho^2, \sigma^4 \rangle$</td>
</tr>
<tr>
<td>4</td>
<td>$n = 1$ and $m = 2$</td>
<td>$\langle \sigma \tau, \rho \rangle$</td>
<td>$\langle \tau^2 \rangle$</td>
</tr>
<tr>
<td></td>
<td>$n = 2$ and $m = 2$</td>
<td>$\langle \sigma \tau, \rho \rangle$</td>
<td>$\langle \tau^2 \rangle$</td>
</tr>
<tr>
<td></td>
<td>$n = 1$ and $m \geq 3$</td>
<td>$\langle \sigma \tau, \rho \rangle$</td>
<td>$\langle (\sigma \tau)^2 \rangle$</td>
</tr>
<tr>
<td></td>
<td>$n = 2$ and $m \geq 3$</td>
<td>$\langle \sigma \tau, \rho, \sigma \rangle$</td>
<td>$\langle \sigma^2 \rho^2 \rangle$</td>
</tr>
<tr>
<td></td>
<td>$m = 2$</td>
<td>$\langle \sigma \rho, \tau \rangle$</td>
<td>$\langle \rho^4 \rangle$</td>
</tr>
<tr>
<td></td>
<td>$m \geq 3$</td>
<td>$\langle \sigma \rho, \tau, \sigma^2 \rangle$</td>
<td>$\langle \rho^2, \sigma^4 \rangle$</td>
</tr>
<tr>
<td>5</td>
<td>$\langle \tau \rho, \sigma \rangle$</td>
<td>$\langle \sigma^2 \rangle$</td>
<td>(2, 2$^{n+1}$)</td>
</tr>
<tr>
<td>6</td>
<td>$\langle \sigma \rho, \tau \rho \rangle$</td>
<td>$\langle \sigma^2 \rho^2 \rangle$</td>
<td>(4, 2$^n$)</td>
</tr>
</tbody>
</table>

Table 2: Subgroups of $G_{1,n}$ of index 4

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$H_{i,4}$</th>
<th>$H'_{i,4}$</th>
<th>$H_{i,4}/H'_{i,4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\langle \sigma, \tau \rangle$</td>
<td>(1)</td>
<td>(2, 2$^n$)</td>
</tr>
<tr>
<td>2</td>
<td>$\langle \tau, \rho \rangle$</td>
<td>$\langle \sigma^2 \rangle$</td>
<td>(2, 4)</td>
</tr>
<tr>
<td>3</td>
<td>$\langle \rho, \sigma \rangle$</td>
<td>$\langle \sigma \rangle$</td>
<td>(2, 4)</td>
</tr>
<tr>
<td>4</td>
<td>$\langle \sigma \tau, \tau \rangle$</td>
<td>(1)</td>
<td>(2, 2$^n$)</td>
</tr>
<tr>
<td>5</td>
<td>$\langle \sigma \rho, \tau \rangle$</td>
<td>$\langle \sigma \rangle$</td>
<td>(2, 4)</td>
</tr>
<tr>
<td>6</td>
<td>$\langle \tau \rho, \sigma \rangle$</td>
<td>$\langle \sigma \rangle$</td>
<td>(2, 4)</td>
</tr>
<tr>
<td>7</td>
<td>$\langle \sigma \tau \rho, \sigma \rangle$</td>
<td>$\langle \sigma \rangle$</td>
<td>(2, 4)</td>
</tr>
</tbody>
</table>

4. $(\sigma \rho)^2 = \rho^2$ and $(\sigma \tau \rho)^2 = (\tau \rho)^2 = \tau^2$.

5. $[\rho, \sigma \tau] = \rho^2 \sigma^2$.

6. $[\rho, \tau^2] = 1$ and for all $r \in \mathbb{N}$, $[\rho, \sigma^{2^r}] = \sigma^{2^{r+1}}$.

Proof: 1., 2., and 3. are obvious, since $\rho^2 = \tau^{2^m} \sigma^{2^n-1}$, $[\rho, \sigma] = \sigma^2$ and $[\rho, \tau] = \rho^2$.

4. $(\sigma \rho)^2 = \sigma \rho \sigma \rho = \sigma \rho^2 \rho^{-1} \sigma \rho = \sigma \rho \sigma^{-1} = \rho^2$.

$(\sigma \tau \rho)^2 = \sigma \tau \rho \sigma \tau \rho = \sigma \tau^2 \rho^{-1} \rho \sigma \rho = \sigma \tau^2 \rho^{-1} \sigma \rho = \sigma \tau \rho \sigma^{-1} = \tau^2$. We proceed similarly to prove the remaining result.

5. Obvious by Lemma 2.1.

6. $[\rho, \tau^2] = \rho^{-1} \tau^{-2} \rho \tau^2 = \rho^{-1} \tau^{-1} \tau^{-1} \rho \tau \tau = \rho^{-1} \tau^{-1} \rho^3 \tau = \rho^{-1} \rho^2 \tau^{-1} \rho \tau = \rho^4 = 1$.

As $[\rho, \tau] = \tau^2$, so $[\rho, \tau^2] = \tau^4$. By induction, we show that for all $r \in \mathbb{N}^*$, $[\rho, \sigma^{2^r}] = \sigma^{2^{r+1}}$.

Let us now prove some entries of the Tables, using Lemmas 2.1 and 2.2.
• For $H_{1,2} = \langle \sigma, \tau, G'_{n,m} \rangle = \langle \sigma, \tau \rangle$, we have $H'_{1,2} = \langle 1 \rangle$, since $[\sigma, \tau] = 1$. As $\sigma^{2^m} = \tau^{2^{m+1}} = 1$, so $H_{1,2}/H'_{1,2} \simeq (2^m, 2^{n+1})$.

• For $H_{2,2} = \langle \sigma, \rho, G'_{n,m} \rangle = \langle \sigma, \rho, \tau^{2^n}, \sigma^2 \rangle = \langle \sigma, \rho, \tau^{2^n} \rangle$. As $\rho^2 = \tau^{2^n} \sigma^{2^{m-1}}$, so $H_{2,2} = \langle \sigma, \rho \rangle$. Therefore, by Lemma 2.2, we get $H'_{2,2} = \langle \sigma^2 \rangle$, thus $H_{2,2}/H'_{2,2} \simeq (2, 4)$, since $\rho^4 = 1$.

• For $H_{4,2} = \langle \sigma \tau, \rho, G'_{n,m} \rangle = \langle \sigma \tau, \rho, \tau^{2^n}, \sigma^2 \rangle = \langle \sigma \tau, \rho, \sigma^2 \rangle = \langle \sigma \tau, \rho, \tau^2 \rangle$, since $\tau^2 = (\sigma \tau)^2 \sigma^{-2}$. We get
  - If $n = 1$ and $m = 2$, then $\rho^2 = \tau^2 \sigma^2$ and $\tau^4 = \sigma^4 = 1$. Lemma 2.2 yields that $H'_{4,2} = \langle \tau^2 \rangle$. Thus $H_{4,2}/H'_{4,2} \simeq (2, 4)$ since $\rho^4 = 1$.
  - If $n \geq 2$ and $m = 2$, then $\rho^2 = \tau^2 \sigma^2$ and $\sigma^4 = 1$, thus $\rho^2 = (\sigma \tau)^2 \sigma^2$; which implies that $\rho^2 (\sigma \tau)^{-2} \sigma^2 = \sigma^2$. Hence $H_{4,2} = \langle \sigma \tau, \rho \rangle$, and Lemma 2.2 yields that $H'_{4,2} = \langle \rho \sigma^2 \rangle$. Thus $H_{4,2}/H'_{4,2} \simeq (4, 2^n)$. If $n = 1$, then $\rho^2 = (\sigma \tau)^2 \sigma^2$; hence $H_{4,2}/H'_{4,2} \simeq (2, 2^{n+1}) = (2, 2^m)$. If $n \neq m - 1$, then $H_{4,2}/H'_{4,2} \simeq (4, 2^{\max(n+1,m)})$.

The other entries of the Tables 1 and 2 are similarly checked.

**Proposition 2.3.** Let $G_{n,m}$ be the group family defined by Formula (1.1), then

1. The order of $G_{n,m}$ is $2^{m+n+2}$ and that of $G'_{n,m}$ is $2^m$.

2. The coclass of $G_{n,m}$ is $n + 2$ and its nilpotency class is $m$.

3. The center, $Z(G)$, of $G$ is of type $(2, 2^m)$.

**Proof:**

1. Since $\sigma^{2^m} = \tau^{2^{m+1}} = 1$ and for all $0 \leq i \leq m - 1$ and $0 \leq j \leq 1$ $\sigma^{2^i} \neq \tau^{2^j}$, then $\langle \sigma, \tau \rangle \simeq (2^m, 2^{n+1})$. Moreover, as $\rho^2 = \tau^{2^n} \sigma^{2^{m-1}}$, so $\langle \sigma, \tau, \rho \rangle \simeq (2^m, 2^{n+1}, 2)$.

2. The lower central series of $G_{n,m}$ is defined inductively by $\gamma_1(G_{n,m}) = G_{n,m}$ and $\gamma_{i+1}(G_{n,m}) = [\gamma_i(G_{n,m}), G_{n,m}]$, that is the subgroup of $G_{n,m}$ generated by the
set \{[a, b] = a^{-1}b^{-1}ab \mid a \in \gamma_i(G_{n,m}), b \in G_{n,m}\}, so the coclass of \(G_{n,m}\) is defined to be \(cc(G_{n,m}) = h - c\), where \(|G_{n,m}| = 2^h\) and \(c = c(G_{n,m})\) is the nilpotency class of \(G_{n,m}\). We easily get
\[
\begin{align*}
\gamma_1(G_{n,m}) &= G_{n,m}, \\
\gamma_2(G_{n,m}) &= G_{n,m} = \langle \sigma^2, \rho^2 \rangle = \langle \sigma^2, \tau^2 \rangle. \\
\gamma_3(G_{n,m}) &= [G', G_{n,m}] = \langle \sigma^4 \rangle \text{ since } [\rho, \tau^2] = 1.
\end{align*}
\]

Then Lemma 2.2(6) yields that for \(j \geq 2\), \(\gamma_{j+1}(G_{n,m}) = [\gamma_j(G_{n,m}), G_{n,m}] = \langle \sigma^{2^j} \rangle\).

Hence \(\gamma_{m+1}(G_{n,m}) = \langle \sigma^{2^m} \rangle = \langle 1 \rangle\) and \(\gamma_m(G_{n,m}) = \langle \sigma^{2^{m-1}} \rangle \neq \langle 1 \rangle\), so \(c(G_{n,m}) = m\). Since \(|G_{n,m}| = 2^{m+n+2}\), then
\[
cc(G_{n,m}) = m + n + 2 - m = n + 2.
\]

3. To prove the last assertion, we use Lemma 12.12 of [13, pp. 204] which states that if \(G\) is a p-group and \(A\) is a normal abelian subgroup of \(G\) such that \(G/A\) is cyclic, then \(A/A \cap Z(G) \cong G'\). Let \(A = H_{1.2}\), so \(A\) is abelian and \([G : A] = 2\), thus \(Z(G) \subseteq A\) and \(A/Z(G) \cong G'\). Hence \(|G| = |A||G : A| = 2|G'||Z(G)|\), thus \(|Z(G)| = \frac{1}{2}|G/G'| = 2^{n+1}\). On the other hand, by Lemma 2.2 we have \([\rho, \sigma^{2^{m-1}}] = \sigma^{2^m} = 1\) and \([\rho, \tau^2] = 1\), so \(\langle \sigma^{2^{m-1}}, \tau^2 \rangle \subseteq Z(G)\). As \(|\langle \sigma^{2^{m-1}}, \tau^2 \rangle| = 2^{n+1}\), so \(\langle \sigma^{2^{m-1}}, \tau^2 \rangle = Z(G) \cong \langle 2, 2^n \rangle\).

We continue with the following results.

**Proposition 2.4** ([14]). Let \(H\) be a normal subgroup of a group \(G\). For \(g \in G\), put \(f = [\langle g \rangle.H : H]\) and let \(\{x_1, x_2, \ldots, x_t\}\) be a set of representatives of \(G/\langle g \rangle.H\). The transfer map \(V_{G \to H} : G/G' \to H/H'\) is given by the following formula
\[
V_{G \to H}(gG') = \prod_{i=1}^{t} x_i^{-1}g^i x_i.H'.
\]

Easily, we prove the following corollaries.

**Corollary 2.5.** Let \(H\) be a subgroup of \(G_{n,m}\) of index 2. If \(G_{n,m}/H = \{1, zH\}\), then
\[
V_{G \to H}(gG_{n,m}) = \begin{cases} 
gz^{-1}gz.H' = g^2[g, z].H' & \text{if } g \in H, \\
g^2.H' & \text{if } g \notin H. \end{cases}
\]

**Corollary 2.6.** Let \(H\) be a normal subgroup of \(G_{n,m}\) of index 4. If \(G_{n,m}/H = \{1, zH, z^2H, z^3H\}\), then
\[
V_{G \to H}(gG_{n,m}) = \begin{cases} 
gz^{-1}gz^{-1}gz^{-1}.H' & \text{if } g \in H, \\
g^4.H' & \text{if } g.H = zH, \\
g^2z^{-1}g^2z.H' & \text{if } g \notin H \text{ and } g.H \neq zH. \end{cases}
\]

**Corollary 2.7.** Let \(H\) be a normal subgroup of \(G_{n,m}\) of index 4. If \(G_{n,m}/H = \{1, z_1H, z_2H, z_3H\}\) with \(z_3 = z_1z_2\), then
\[
V_{G \to H}(gG_{n,m}) = \begin{cases} 
 gz_1^{-1}gz_1^{-1}gz_1^{-1}g^{-1}gz_1z_2.H' & \text{if } g \in H, \\
g^2z_1^{-1}g^2z_1.H' & \text{if } g.H = z_3H \text{ with } i \neq j. \end{cases}
\]
We can now establish our main result. Let \( \ker V_H \) denote the kernel of the transfer map \( V_{G_{n,m}}: G_{n,m}/G_{n,m}' \to H/H' \), where \( H \) is a subgroup of \( G_{n,m} \).

**Theorem 2.8.** Keep the previous notations. Then

1. \( \ker V_{H_{1,2}} = \langle \sigma G_{n,m} \rangle \).
2. \( \ker V_{H_{2,2}} = \langle \sigma G_{n,m}', \rho G_{n,m} \rangle \).
3. \( \ker V_{H_{3,2}} = \begin{cases} \langle \tau \rho G_{n,m}, \sigma G_{n,m}' \rangle & \text{if } m = 2 \text{ and } n = 1, \\ \langle \tau G_{n,m}, \sigma G_{n,m}' \rangle & \text{if } m \geq 3 \text{ and } n = 1, \\ \langle \sigma G_{n,m}' \rangle & \text{otherwise} . \end{cases} \)
4. \( \ker V_{H_{4,2}} = \begin{cases} \langle \tau G_{n,m}, \rho G_{n,m}' \rangle & \text{if } m = 2 \text{ and } n = 1, \\ \langle \sigma G_{n,m}', \sigma G_{n,m} \rangle & \text{if } m \geq 3 \text{ and } n = 1, \\ \langle \rho G_{n,m}' \rangle & \text{otherwise} . \end{cases} \)
5. \( \ker V_{H_{5,2}} = \begin{cases} \langle \rho G_{n,m}', \tau G_{n,m}' \rangle & \text{if } m = 2 \text{ and } n = 1, \\ \langle \rho G_{n,m}', \tau G_{n,m} \rangle & \text{if } m \geq 3 \text{ and } n = 1, \\ \langle \rho G_{n,m}' \rangle & \text{otherwise} . \end{cases} \)
6. \( \ker V_{H_{6,2}} = \begin{cases} \langle \tau G_{n,m}', \sigma G_{n,m} \rangle & \text{if } n = 1, \\ \langle \sigma G_{n,m} \rangle & \text{if } n \geq 2. \end{cases} \)
7. \( \ker V_{H_{7,2}} = \begin{cases} \langle \sigma G_{n,m}', \tau G_{n,m}' \rangle & \text{if } m = 2 \text{ and } n = 1, \\ \langle \sigma G_{n,m}', \tau G_{n,m} \rangle & \text{if } m \geq 3 \text{ and } n = 1, \\ \langle \sigma G_{n,m}' \rangle & \text{otherwise} . \end{cases} \)
8. If \( n = 1 \), then for all \( 1 \leq i \leq 7 \), \( \ker V_{H_{i,4}} = G_{1,m}/G_{1,m}' \).

**Proof:** We prove only some assertions, the others are similarly shown.

1. We know, from the Table 1, that \( H_{1,2} = \langle \sigma, \tau \rangle \), then \( G_{m,n}/H_{1,2} = \{1, \rho H_{1,2}\} \) and \( H_{1,2} = \langle 1 \rangle \). Hence, by Corollary 2.5 and Lemma 2.2, we get

   * \( V_{G_{m,n} \to H_{1,2}}(\sigma G_{m,n}') = \sigma^2[\sigma, \rho] H_{1,2}' = \sigma^2\sigma^{-2} H_{1,2}' = H_{1,2}' \).
   * \( V_{G_{m,n} \to H_{1,2}}(\tau G_{m,n}') = \tau^2[\tau, \rho] H_{1,2}' = \tau^2\rho^{-2} H_{1,2}' = \tau^2\rho^2 H_{1,2}' \neq H_{1,2}' \).
   * \( V_{G_{m,n} \to H_{1,2}}(\rho G_{m,n}') = \rho^2 H_{1,2}' \neq H_{1,2}' \).
   * \( V_{G_{m,n} \to H_{1,2}}(\sigma \tau G_{m,n}') = (\sigma \tau)^2[\sigma \tau, \rho] H_{1,2}' = (\sigma \tau)^2\sigma^{-2} \rho^2 H_{1,2}' = \tau^2\rho^2 H_{1,2}' \neq H_{1,2}' \).
   * \( V_{G_{m,n} \to H_{1,2}}(\sigma^{'} \rho G_{m,n}') = (\sigma^{'})^2[\sigma^{'}, \rho^{'}] H_{1,2}' = \rho^2 H_{1,2}' \neq H_{1,2}' \).
   * \( V_{G_{m,n} \to H_{1,2}}(\tau^{'})^2[\tau^{'}, \rho^{'}] H_{1,2}' = \tau^2 H_{1,2}' \neq H_{1,2}' \).
   * \( V_{G_{m,n} \to H_{1,2}}(\sigma \tau \rho G_{m,n}') = (\sigma \tau \rho)^2 H_{1,2}' = \tau^2 H_{1,2}' \neq H_{1,2}' \).
Therefore ker $V_{H,2} = \langle \sigma G^\prime_{m,n} \rangle$.

3. Similarly, from the Table 1, we get $H_{3,2} = \begin{cases} \langle \tau, \rho \rangle & \text{if } m = 2 , \\ \langle \tau, \rho, \sigma^2 \rangle & \text{if } m \geq 3. \end{cases}$ Then

$G_{m,n}/H_{3,2} = \{ 1, \sigma H_{3,2} \}$ and $H'_{3,2} = \begin{cases} \langle \rho^2 \rangle = \langle (\sigma \tau)^2 \rangle & \text{if } m = 2 \text{ and } n = 1 , \\ \langle \rho^2 \rangle = \langle \sigma^2 \tau^2 \rangle & \text{if } m = 2 \text{ and } n \geq 2 , \\ \langle \rho^2, \sigma^4 \rangle = \langle \sigma^4, \tau^2 \rangle & \text{if } m \geq 3 \text{ and } n = 1 , \\ \langle \rho^2, \sigma^4 \rangle = \langle \sigma^4, \tau^2 \rangle & \text{if } m \geq 3 \text{ and } n \geq 2. \end{cases}$

Hence, by Corollary 2.5 and Lemma 2.2, we get

1st case: $m = 2$.

- $V_{G_{m,n}, H_{3,2}}(\sigma G^\prime_{m,n}) = \sigma^2 H'_{3,2} \neq H'_{3,2}$.
- $V_{G_{m,n}, H_{3,2}}(\tau G^\prime_{m,n}) = \tau^2 = H'_{3,2}$.
- $V_{G_{m,n}, H_{3,2}}(\rho G^\prime_{m,n}) = \rho^2 = H'_{3,2}$.
- $V_{G_{m,n}, H_{3,2}}(\sigma \rho G^\prime_{m,n}) = \sigma^2 \tau \rho H'_{3,2} \neq H'_{3,2}$.

Therefore ker $V_{H_{3,2}} = \begin{cases} \langle \tau \rho G^\prime_{m,n}, \sigma \rho G^\prime_{m,n} \rangle & \text{if } m = 2 \text{ and } n = 1 , \\ \langle \sigma \rho G^\prime_{m,n} \rangle & \text{if } m = 2 \text{ and } n \geq 2. \end{cases}$

2nd case: $m \geq 3$.

- $V_{G_{m,n}, H_{3,2}}(\sigma G^\prime_{m,n}) = \sigma^2 H'_{3,2} \neq H'_{3,2}$.
- $V_{G_{m,n}, H_{3,2}}(\tau G^\prime_{m,n}) = \tau^2 = H'_{3,2}$.
- $V_{G_{m,n}, H_{3,2}}(\rho G^\prime_{m,n}) = \rho^2 = H'_{3,2}$.
- $V_{G_{m,n}, H_{3,2}}(\sigma \rho G^\prime_{m,n}) = \sigma^2 \tau \rho H'_{3,2} \neq H'_{3,2}$.

Therefore, ker $V_{H'_{3,2}} = \begin{cases} \langle \sigma \rho G^\prime_{m,n}, \tau G^\prime_{m,n} \rangle & \text{if } m \geq 3 \text{ and } n = 1 , \\ \langle \sigma \rho G^\prime_{m,n} \rangle & \text{if } m \geq 3 \text{ and } n \geq 2. \end{cases}$

8. We know, from the Table 2, that $H_{1,4} = \langle \sigma, \tau^2 \rangle$, then $G_{n,m}/H_{1,4} = \{ 1, \tau H_{1,4}, \rho H_{1,4}, \tau \rho H_{1,4} \}$ and $H'_{1,4} = \{ 1 \}$. Hence Corollary 2.7 and Lemma 2.2 yield that

- $V_{G_{n,m}, H_{1,4}}(\sigma G^\prime_{m,n}) = \sigma^2 \rho H'_{1,4} = \sigma^2 \rho^2 H'_{1,4} = H'_{1,4}$.
- $V_{G_{n,m}, H_{1,4}}(\tau G^\prime_{m,n}) = \tau^2 \rho^2 = \tau^2 \rho H'_{1,4} = \tau^4 H'_{1,4} = H'_{1,4}$.
- $V_{G_{n,m}, H_{1,4}}(\rho G^\prime_{m,n}) = \rho^2 \tau = \rho^2 \tau H'_{1,4} = \rho^4 \rho^2 H'_{1,4} = H'_{1,4}$.

Therefore, ker $V_{H_{1,4}} = \langle \sigma G^\prime_{m,n}, \tau G^\prime_{m,n}, \rho G^\prime_{m,n} \rangle = G_{n,m}/G^\prime_{m,n}$. \(\square\)
3. Applications

Let $k$ be a number field and $C_{k,2}$ be its 2-class group, that is the 2-Sylow subgroup of the ideal class group $C_k$ of $k$, in the wide sens. Let $k_2^{(1)}$ be the Hilbert 2-class field of $k$ in the wide sens. Then the Hilbert 2-class field tower of $k$ is defined inductively by: $k_2^{(0)} = k$ and $k_2^{(2^j)} = (k_2^{(2^{j-1})})^{2^i}$, where $i$ is a positive integer. Let $M$ be an unramified extension of $k$ and $C_{k,M}$ be the subgroup of $C_k$ associated to the class of an ideal $A$ of $k$ the class of the ideal generated by $A$ in $M$, and by $N_{M/k}$ the norm of the extension $M/k$.

Throughout all this section, assume that $\text{Gal}(k_2^{(2)}/k) \cong G_{n,m}$. Hence, according to Class Field Theory, $C_{k,2} \cong G_{n,m}/G'_{n,m} \cong (2, 2, 2^n)$, thus $C_{k,2} = \langle a, b, c \rangle \cong \langle \sigma_{G_{n,m}}, \tau_{G_{n,m}}, \rho_{G_{n,m}} \rangle$, where $\langle a, k_2^{(2)}/k \rangle = \sigma_{G_{n,m}}$, $\langle b, k_2^{(2)}/k \rangle = \tau_{G_{n,m}}$ and $(c, k_2^{(2)}/k) = \rho_{G_{n,m}}$, with $(\ldots, k_2^{(2)}/k)$ denotes the Artin symbol in $k_2^{(2)}/k$.

It is well known that each subgroup $H_{i,j}$, where $1 \leq i \leq 7$ and $j = 2$ or 4, of $C_{k,2}$ is associated, by class field theory, to a unique unramified extension $K_{i,j}$ of $k_2^{(1)}$ such that $H_{i,j}/H_{i,j}' \cong C_{k_{i,j},2}$.

Our goal is to study the capitulation problem of the 2-ideal classes of $k$ in its unramified quadratic extensions $K_{i,2}$ and in its unramified biquadratic extensions $K_{i,4}$ if $n = 1$. By Class Field Theory, the kernel of $j_{k\rightarrow M}$, $\ker j_{k\rightarrow M}$, is determined by the kernel of the transfer map $V_{G\rightarrow H} : G/G' \rightarrow H/H'$, where $G = \text{Gal}(k_2^{(2)}/k)$ and $H = \text{Gal}(k_2^{(2)}/M)$.

**Theorem 3.1.** Keep the previous notations.

1. $\ker j_{k\rightarrow K_{1,2}} = \langle a \rangle$.
2. $\ker j_{k\rightarrow K_{2,2}} = \langle a, c \rangle$
3. $\ker j_{k\rightarrow K_{3,2}} = \langle b, c \rangle$ if $m = 2$ and $n = 1$,
   $\langle b, ac \rangle$ if $m \geq 3$ and $n = 1$,
   $\langle ac \rangle$ otherwise.
4. $\ker j_{k\rightarrow K_{4,2}} = \langle b, c \rangle$ if $m = 2$ and $n = 1$,
   $\langle ab, c \rangle$ if $m \geq 3$ and $n = 1$,
   $\langle c \rangle$ otherwise.
5. $\ker j_{k\rightarrow K_{5,2}} = \langle c, ab \rangle$ if $m = 2$ and $n = 1$,
   $\langle c, b \rangle$ if $m \geq 3$ and $n = 1$,
   $\langle c \rangle$ otherwise.
6. $\ker j_{k\rightarrow K_{6,2}} = \langle b, c \rangle$ if $n = 1$,
   $\langle ac \rangle$ if $n \geq 2$.
7. $\ker j_{k\rightarrow K_{7,2}} = \langle ac, b \rangle$ if $m = 2$ and $n = 1$,
   $\langle ac, bc \rangle$ if $m \geq 3$ and $n = 1$,
   $\langle ac \rangle$ otherwise.
8. If \( n = 1 \), then for all \( 1 \leq i \leq 7 \), \( \ker j_{k \to K_{i,4}} = C_{k,2} \).

9. The \( 2 \)-class group of \( k^{(2)}_2 \) is of type \( (2, 2^{m-1}) \).

10. The Hilbert \( 2 \)-class field tower of \( k \) stops at \( k^{(2)}_2 \).

**Proof:** According to the Theorem 2.8, we have

1. Since \( \ker V_{H_{1,2}} = \langle \sigma G'_{n,m} \rangle \), so \( \ker j_{k \to K_{1,2}} = \langle a \rangle \).

2. As \( \ker V_{H_{2,2}} = \langle \sigma G'_{n,m}, \rho G'_{n,m} \rangle \), \( j_{k \to K_{2,2}} = \langle a, c \rangle \).

3. Similarly, as \( \ker V_{H_{3,2}} = \begin{cases} \langle \tau p G'_{n,m}, \sigma G'_{n,m} \rangle & \text{if } m = 2 \text{ and } n = 1, \\ \langle \tau G'_{n,m}, \sigma G'_{n,m} \rangle & \text{if } m \geq 3 \text{ and } n = 1, \\ \langle \sigma G'_{n,m} \rangle & \text{otherwise} \end{cases} \).

Then \( \ker j_{k \to K_{3,2}} = \begin{cases} \langle bc, ac \rangle & \text{if } m = 2 \text{ and } n = 1, \\ \langle bc, ac \rangle & \text{if } m \geq 3 \text{ and } n = 1, \\ \langle ac \rangle & \text{otherwise} \end{cases} \).

The other assertions are similarly proved.

8. It is well known, by class field theory, that \( C_{K^{(1)},2} \simeq G_{n,m}^{(1)} \), where \( C_{k^{(1)},2} \) is the \( 2 \)-class group of \( k^{(1)}_2 \). As \( G_{n,m}^{(1)} = \langle 2^d, \tau^{2^n} \rangle \simeq (2, 2^{m-1}) \), since \( \sigma^{2n} = \tau^4 = 1 \). So the result.

9. For every \( n \geq 1 \), we have \( H_{1,4} = \langle \sigma, G'_{n,m} \rangle = \langle \sigma, \tau^{2^n} \rangle \simeq (2, 2^n) \), \( H_{2,4} = \langle \tau, \sigma^2 \rangle \simeq (2^n, 2^{m-1}) \) and \( H_{3,4} = \langle \sigma, G'_{n,m} \rangle = \langle \sigma, \tau^{2^n} \rangle \simeq (2^{\min(m-1, n)}, 2^{\max(m, n+1)}) \) are the three subgroups of index 2 of the group \( H_{1,2} \), then \( K_{1,4}, K_{2,4} \) and \( K_{3,4} \) are the three unramified quadratic extensions of \( K_{1,2} \). On the other hand, the 2-class groups of these fields are of rank 2, since, by Class Field Theory, \( C_{K_{i,j},2} \simeq H_{i,j}/H_{i,j} \) with \( i = 1, 2 \) or 4 and \( j = 2 \) or 4. Thus \( C_{K_{i,j},2} \simeq (2^n, 2) \) and \( C_{K_{2,2}} \simeq (2^n, 2^{n+1}) \). Hence \( h_2(K_{2,4}) = \frac{h_2(K_{2,4})}{2} \), where \( h_2(K) \) denotes the 2-class number of the field \( K \). Therefore, we can apply Proposition 7 of [8], which says that \( K_{1,2} \) has an abelian 2-class field tower if and only if it has a quadratic unramified extension \( K_{2,4}/K_{1,2} \) such that \( h_2(K_{2,4}) = \frac{h_2(K_{1,2})}{2} \). Thus \( K_{1,2} \) has abelian 2-class field tower which terminates at the first stage; this implies that the 2-class field tower of \( k \) terminates at \( k^{(2)}_2 \), since \( k \subset K_{1,2} \). Moreover, we know, from Proposition 2.3, that \( |G_{n,m}| = 2^{m+n+2} \) and \( |G'_{n,m}| = 2^m \), hence \( k^{(1)}_2 \neq k^{(2)}_2 \).

\[ \square \]

4. Example

Let \( k = \mathbb{Q}(\sqrt{d}) \) be an imaginary quadratic number field with discriminant \( d = -4pqq' \), where \( p \equiv 5 \mod 8 \), \( q \equiv 3 \mod 8 \) and \( q' \equiv 7 \mod 8 \) are primes such that \( \left( \frac{q}{p} \right) = \left( \frac{q'}{p} \right) = -1 \). Let \( k^{(1)}_2 \) be the Hilbert 2-class field of \( k \), \( k^{(2)}_2 \) its second Hilbert 2-class field and \( G \) be the Galois group of \( k^{(2)}_2/k \). According to [10], \( k \) has an elementary abelian 2-class group \( C_{k,2} \) of rank 3, that is of type \( (2, 2, 2) \). Denote by \( h_2(-qq') \) the 2-class number of \( \mathbb{Q}(\sqrt{-qq'}) \), then by [15,10] \( h_2(-qq') = 2^m \) and the 2-class group of \( \mathbb{Q}(\sqrt{-qq'}) \) is of type \( (2, 2^{m-1}) \) with \( m \geq 2 \). By [10, Theorem 1], we have \( G \simeq G_{1,m} \). As \( C_{k,2} \simeq (2, 2, 2) \), then \( k \) has seven unramified quadratic
extensions and seven unramified biquadratic extensions within his first Hilbert 2-class field $k_2^{(1)}$. For more details about the results given in this section and about the following theorem the reader can see [10]. This theorem is given here to illustrate the results shown in the above sections.

**Theorem 4.1.** Let $k = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic number field with discriminant $d = -4pqq'$, where $p \equiv 5 \mod 8$, $q \equiv 3 \mod 8$ and $q' \equiv 7 \mod 8$ are primes such that $\left(\frac{d}{p}\right) = \left(\frac{d}{q}\right) = -1$. $k$ has fourteen unramified extensions within his first Hilbert 2-class field, $k_2^{(1)}$. Denote by $C_{k,2}$ the 2-class group of $k$. Then the following assertions hold.

1. $C_{k,2}$ is of type $(2, 2, 2)$.
2. Exactly four elements of $C_{k,2}$ capitulate in each unramified quadratic extension of $k$ except one where only 2 classes capitulate.
3. All the 2-classes of $k$ capitulate in each unramified biquadratic extension of $k$.
4. The Hilbert 2-class field tower of $k$ stops at $k_2^{(2)}$.
5. $C_{k_2^{(1)},2} \simeq (2, 2^{m-1})$.
6. The coclass of $G$ is 3 and its nilpotency class is $m$.
7. The 2-class groups of the unramified quadratic extensions of $k$ are of types $(2, 4)$, $(2, 2, 2)$ or $(4, 2^m)$.
8. The 2-class groups of the unramified biquadratic extensions of $k$ are of types $(2, 4)$, $(2, 2^m)$ or $(4, 2^{m-1})$.

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