Generalized derivations in prime and semiprime rings

Shuliang Huang and Nadeem ur Rehman

ABSTRACT: Let $R$ be a prime ring, $I$ a nonzero ideal of $R$ and $m, n$ fixed positive integers. If $R$ admits a generalized derivation $F$ associated with a nonzero derivation $d$ such that $(F([x, y])^m) = [x, y]^n$ for all $x, y \in I$, then $R$ is commutative. Moreover we also examine the case when $R$ is a semiprime ring.

Key Words: prime and semiprime rings, generalized derivations, GPIs.

Contents

1 Introduction 29
2 The case: $R$ a prime ring 30
3 The case: $R$ a semiprime ring 32

1. Introduction

In all that follows, unless stated otherwise, $R$ will be an associative ring, $Z(R)$ the center of $R$, $Q$ its Martindale quotient ring and $U$ its Utumi quotient ring. The center of $U$, denoted by $C$, is called the extended centroid of $R$ (we refer the reader to [3] for these objects). For any $x, y \in R$, the symbol $[x, y]$ and $x \circ y$ stand for the commutator $xy - yx$ and anti-commutator $xy + yx$, respectively. For each $x, y \in R$ and each $n \geq 1$, define $[x, y]_1 = xy - yx$ and $[x, y]_k = [[x, y]_{k-1}, y]$ for $k \geq 2$. Recall that a ring $R$ is prime if for any $a, b \in R$, $aRb = (0)$ implies $a = 0$ or $b = 0$, and is semiprime if for any $a \in R$, $aRa = (0)$ implies $a = 0$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. In [4], Bresar introduced the definition of generalized derivation: an additive mapping $F : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$, and $d$ is called the associated derivation of $F$. Hence, the concept of generalized derivations covers both the concepts of a derivation and of a left multiplier. Basic examples are derivations and generalized inner derivations. We refer to call such mappings generalized inner derivations for the reason they present a generalization of the concept of inner derivations. In [9], Hvala studied generalized derivations in the context of algebras on certain norm spaces. In [13], Lee extended the definition of a generalized derivation as follows: by a generalized derivation we mean an additive mapping $F : I \to U$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in I$, where $I$ is a dense left ideal of $R$ and $d$ is a derivation from $I$ into $U$. Moreover, Lee also proved that every generalized derivation can be uniquely extended to a
generalized derivation of \( U \) and thus all generalized derivations of \( R \) will be implicitly assumed to be defined on the whole of \( U \). Lee obtained the following: every generalized derivation \( F \) on a dense left ideal of \( R \) can be uniquely extended to \( U \) and assumes the form \( F(x) = ax + d(x) \) for some \( a \in U \) and a derivation \( d \) on \( U \).

This paper is included in a line of investigation concerning the relationship between the structure of a ring \( R \) and the behaviour of some additive mappings defined on \( R \) satisfy certain special identities. In [1], Ashraf and Rehman proved that if \( R \) is a prime ring, \( I \) a nonzero ideal of \( R \) and \( d \) is a derivation of \( R \) such that \( d(x \circ y) = x \circ y \) for all \( x, y \in I \), then \( R \) is commutative. In [2, Theorem 1], Argac and Inceboz generalized the above result as following: Let \( R \) be a prime ring, \( I \) a nonzero ideal of \( R \) and \( n \) a fixed positive integer, if \( R \) admits a derivation \( d \) with the property \( (d(x \circ y))^n = x \circ y \) for all \( x, y \in I \), then \( R \) is commutative. In [7], Daif and Bell showed that if in a semiprime ring \( R \) there exists a nonzero ideal \( I \) of \( R \) and a derivation \( d \) such that \( d([x, y]) = [x, y] \) for all \( x, y \in I \), then \( I \subseteq Z(R) \). At this point the natural question is what happens in case the derivation is replaced by a generalized derivation. In [18], Quadri et al., proved that if \( R \) is a prime ring, \( I \) a nonzero ideal of \( R \) and \( F \) a generalized derivation associated with a nonzero derivation \( d \) such that \( F([x, y]) = [x, y] \) for all \( x, y \in I \), then \( R \) is commutative. In [10], we studied a similar condition and proved that a prime ring \( R \) satisfying \( (F(x \circ y))^n = x \circ y \) must be commutative. The present paper is motivated by the previous results and we here continue this line of investigation by examining what happens a ring \( R \) satisfying the identity \( (F([x, y])^m = [x, y]^m \). Explicitly we shall prove the following:

**Theorem 1.1.** Let \( R \) be a prime ring, \( I \) a nonzero ideal of \( R \) and \( m, n \) fixed positive integers. If \( R \) admits a generalized derivation \( F \) associated with a nonzero derivation \( d \) such that \( (F([x, y])^m = [x, y]^m \) for all \( x, y \in I \), then \( R \) is commutative.

**Theorem 1.2.** Let \( R \) be a semiprime ring and \( m, n \) fixed positive integers. If \( R \) admits a generalized derivation \( F \) associated with a derivation \( d \) such that \( (F([x, y])^m = [x, y]^m \) for all \( x, y \in R \), then there exists a central idempotent element \( e \) in \( U \) such that on the direct sum decomposition \( R = eU \oplus (1-e)U \), \( d \) vanishes identically on \( eU \) and the ring \( (1-e)U \) is commutative.

2. The case: \( R \) a prime ring

**Theorem 2.1.** Let \( R \) be a prime ring, \( I \) a nonzero ideal of \( R \) and \( m, n \) fixed positive integers. If \( R \) admits a generalized derivation \( F \) associated with a nonzero derivation \( d \) such that \( (F([x, y])^m = [x, y]^m \) for all \( x, y \in I \), then \( R \) is commutative.

**Proof:** Since \( R \) is a prime ring and \( F \) is a generalized derivation of \( R \), by Lee [13, Theorem 3], \( F(x) = ax + d(x) \) for some \( a \in U \) and a derivation \( d \) on \( U \). By the given hypothesis we have now \( [x, y]^m = (a[x, y] + d([x, y]))^m = (a[x, y] + d(x), y) + [x, d(y)]^m \) for all \( x, y \in I \). By Kharchenko [12], we divide the proof into two cases:
Case 1. Let \( d \) be an outer derivation of \( U \), then \( I \) satisfies the polynomial identity 
\( (a[x, y] + [s, t] + [x, t])^n = [x, y]^n \) for all \( x, y, s, t \in I \). In particular, for \( y = 0 \), \( I \) satisfies the blended component \( ([x, t])^n = 0 \) for all \( x, t \in I \), by Herstein [11, Theorem 2], we have \( I \subseteq Z(R) \), and so \( R \) is commutative by Mayne [17, Lemma 3].

Case 2. Let now \( d \) be the inner derivation induced by an element \( q \in Q \), that is \( d(x) = [q, x] \) for all \( x, y \in U \). It follows that \( (a[x, y] + [q, y])^n = [x, y]^n \) for all \( x, y \in I \). By Chuang [5, Theorem 2], \( I \) and \( Q \) satisfy the same generalized polynomial identities (GPIs), we have \( (a[x, y] + [q, y])^n = [x, y]^n \) for all \( x, y \in Q \). In case center \( C \) of \( Q \) is infinite, we have \( (a[x, y] + [q, y])^n = [x, y]^n \) for all \( x, y \in Q \otimes Q, C \), where \( C \) is the algebraic closure of \( C \). Since both \( Q \) and \( Q \otimes Q, C \) are prime and centrally closed [8, Theorem 2.5 and Theorem 3.5], we may replace \( R \) by \( Q \otimes Q, C \) according as \( C \) is finite or infinite. Thus we may assume that \( R \) is centrally closed over \( C \) (i.e. \( RC = C \)) which is either finite or algebraically closed and \( (a[x, y] + [q, y])^n = [x, y]^n \) for all \( x, y \in R \). By Martindale [16, Theorem 3], \( RC \) (and so \( R \)) is a primitive ring which is isomorphic to a dense ring of linear transformations of a vector space \( V \) over a division ring \( D \).

Assume that \( \dim V_D \geq 3 \).

First of all, we want to show that \( v \) and \( qv \) are linearly \( D \)-dependent for all \( v \in V \). Since if \( qv = 0 \) then \( v, qv \) is \( D \)-dependent, suppose that \( qv \neq 0 \). If \( v \) and \( qv \) are \( D \)-independent, since \( \dim V_D \geq 3 \), then there exists \( w \in V \) such that \( v, qv, w \) are also \( D \)-independent. By the density of \( R \), there exists \( x, y \in R \) such that: \( xv = 0, xqv = w, xw = v; yv = 0, yyv = 0, yw = v \). These imply that \( v = (a[x, y] + [q, y])^n v = [x, y]^n v = 0 \), which is a contradiction. So we conclude that \( v \) and \( qv \) are linearly \( D \)-dependent for all \( v \in V \).

Our next goal is to show that there exists \( b \in D \) such that \( qv = vb \) for all \( v \in V \). In fact, choose \( v, w \in V \) linearly independent. Since \( \dim V_D \geq 3 \), there exists \( u \in V \) such that \( u, v, w \) are linearly independent, and so \( b_u, b_v, b_w \in D \) such that \( qu = ub_u, qv = vb_v, qw = wb_w \), that is \( q(u + v + w) = ub_u + vb_v + wb_w \). Moreover \( q(u + v + w) = (u + v + w)b_{u+v+w} \) for a suitable \( b_{u+v+w} \in D \). Then \( 0 = u(b_{u+v+w} - b_u) + v(b_{u+v+w} - b_v) + w(b_{u+v+w} - b_w) \) and because \( u, v, w \) are linearly independent, \( b_u = b_v = b_w = b_{u+v+w} \), that is \( b \) does not depend on the choice of \( v \). Hence now we have \( qv = vb \) for all \( v \in V \).

Now for \( r \in R, v \in V \), we have \( (rq)v = r(qv) = r(vb) = (rv)b = q(rv), \) that is \([q, R]V = 0 \). Since \( V \) is a left faithful irreducible \( R \)-module, hence \([q, R] = 0 \), i.e. \( q \in Z(R) \) and so \( d = 0 \), a contradiction.

Suppose now that \( \dim V_D \leq 2 \).

In this case \( R \) is a simple GPI-ring with 1, and so it is a central simple algebra
finite dimensional over its center. By Lanski [14, Lemma 2], it follows that there exists a suitable filed \( F \) such that \( R \subseteq M_k(F) \), the ring of all \( k \times k \) matrices over \( F \), and moreover \( M_k(F) \) satisfies the same GPI as \( R \).

Assume \( k \geq 3 \), by the same argument as in the above, we can get a contradiction.

Obviously if \( k = 1 \), then \( R \) is commutative.
Thus we may assume that \( k = 2 \) i.e., \( R \subseteq M_2(F) \), where \( M_2(F) \) satisfies 
\[ (a[x, y] + [[y, x], y] + [x, [y, y]])^m = [x, y]^n. \]
Denote \( e_{ij} \) the usual matrix unit with 1 in \((i, j)\)-entry and zero elsewhere. 
Let \( [x, y] = [e_{21}, e_{11}] = e_{21} \). Then \([x, y]^n = e_{21} \). In this case we have \((ae_{21} + qe_{21} - e_{21}q)^m = e_{21} \). Right multiplying by \( e_{21} \), we get \((-1)^m(e_{21}q)^m e_{21} = (ae_{21} + qe_{21} - e_{21}q)^m e_{21} = e_{21}e_{21} = 0 \). Set \( q = \left( \begin{array}{cc} q_{11} & q_{12} \\ q_{21} & q_{22} \end{array} \right) \). By calculation we find
\[ (-1)^m \left( \begin{array}{cc} 0 & 0 \\ q_{12} & 0 \end{array} \right) = 0, \]
which implies that \( q_{12} = 0 \). Similarly we can see that \( q_{21} = 0 \). Therefore \( q \) is diagonal in \( M_2(F) \). Let \( f \in \text{Aut}(M_2(F)) \). Since 
\[ f(a)[f(x), f(y)] + [[f(q), f(x)], f(y)] + [f(x), [f(q), f(y)]]^m = [f(x), f(y)]^n, \]
so \( f(q) \) must be a diagonal matrix in \( M_2(F) \). In particular, let \( f(x) = (1 - e_{ij})x(1 + e_{ij}) \)
for \( i \neq j \), then \( f(q) = q + (q_{ii} - q_{ij})e_{ij} \), that is \( q_{ii} = q_{ij} \) for \( i \neq j \). This implies that \( q \) is central in \( M_2(F) \), which leads to \( d = 0 \), a contradiction. This completes the proof of the theorem. \( \square \)

The following example demonstrates that \( R \) to be prime is essential in the hypothesis.

**Example 2.2.** Consider \( S \) be any ring and let \( R = \left\{ \left( \begin{array}{cc} a & b \\ 0 & 0 \end{array} \right) \mid a, b \in S \right\} \) and let 
\[ I = \left\{ \left( \begin{array}{cc} 0 & a \\ 0 & 0 \end{array} \right) \mid a \in S \right\} \]
be a nonzero ideal of \( R \). We define a map \( F : R \rightarrow R \) by \( F(x) = 2e_{11}x - xe_{11} \). Then it is easy to see that \( F \) is a generalized derivation associated with a nonzero derivation \( d(x) = e_{11}, x \). It is straightforward to check that \( F \) satisfies the property: \( (F([x, y]))^m = [x, y]^n \) for all \( x, y \in I \). However, \( R \) is not commutative.

### 3. The case: \( R \) a semiprime ring

**Theorem 3.1.** Let \( R \) be a semiprime ring and \( m, n \) fixed positive integers. If \( R \) admits a generalized derivation \( F \) associated with a derivation \( d \) such that 
\[ (F([x, y]))^m = [x, y]^n \]
for all \( x, y \in R \), then there exists a central idempotent element \( e \) in \( U \) such that on the direct sum decomposition \( R = eU \oplus (1 - e)U \), \( d \) vanishes identically on \( eU \) and the ring \((1 - e)U\) is commutative.

**Proof:** Since \( R \) is semiprime and \( F \) is a generalized derivation of \( R \), by Lee [13, Theorem 3], \( F(x) = ax + d(x) \) for some \( a \in U \) and a derivation \( d \) on \( U \). We are given that \( (a[x, y] + d([x, y]))^m = [x, y]^n \) for all \( x, y \in R \). By Lee [15, Theorem 3], \( R \) and \( U \) satisfy the same differential identities, then \( (a[x, y] + d([x, y]))^m = [x, y]^n \) for all \( x, y \in U \). Let \( B \) be the complete Boolean algebra of idempotents in \( C \) and \( M \) be any maximal ideal of \( B \). Since \( U \) is a \( B \)-algebra orthogonal complete [6, p.42] and \( MU \) is a prime ideal of \( U \), which is \( d \)-invariant. Denote \( \overline{U} = U/MU \) and \( \overline{d} \) the derivation induced by \( d \) on \( \overline{U} \), i.e., \( \overline{d(u)} = \overline{d(u)} \) for all \( u \in U \). For all \( \overline{x}, \overline{y} \in \overline{U} \), \( (\overline{d([x, y]))^m = [\overline{x}, \overline{y}]^n \). It is obvious that \( \overline{U} \) is prime. Therefore by
Theorem 2.1, we have either $U$ is commutative or $d = 0$, that is either $d(U) \subseteq MU$ or $[U, U] \subseteq MU$. Hence $d(U)[U, U] \subseteq MU$, where $MU$ runs over all prime ideals of $U$. Since $\cap_M MU = 0$, we obtain $d(U)[U, U] = 0$.

By using the theory of orthogonal completion for semiprime rings (see [3, Chapter 3]), it is clear that there exists a central idempotent element $e$ in $U$ such that on the direct sum decomposition $R = eU \oplus (1 - e)U$, $d$ vanishes identically on $eU$ and the ring $(1 - e)U$ is commutative. This completes the proof of the theorem.

\begin{flushright}
$\blacksquare$
\end{flushright}

Acknowledgments

This research work is supported by the Anhui Provincial Natural Science Foundation (1408085QA08) and the key University Science Research Project of Anhui Province (KJ2014A183) of China.

References

11. I. N. Herstein, Center-like elements in prime rings, J. Algebra, 60(1979), 567-574.


Shuliang Huang
Department of Mathematics
Chuzhou University, Chuzhou Anhui
239012, P. R. CHINA
E-mail address: shulianghuang@163.com

and

Nadeem ur Rehman
Department of Mathematics
Aligarh Muslim University
Aligarh 202002, INDIA
E-mail address: rehman100@gmail.com